Homework #1
MTH 106: Fall 2025
Instructor: John Gemmer

Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a "rough draft"; your submission with well-organized calculations and relevant explanations should be thought of as your "final draft".

Problems to be completed by all students

Problem 1. Consider the system $\dot{x} = \sin(x)$.

(a) Find all fixed points of the flow.

(b) At which points x does the flow have the greatest velocity to the right?

(c) Find the flows acceleration $\ddot{x}$ as a function of x.

(d) Find the points where the flow has maximum positive acceleration.

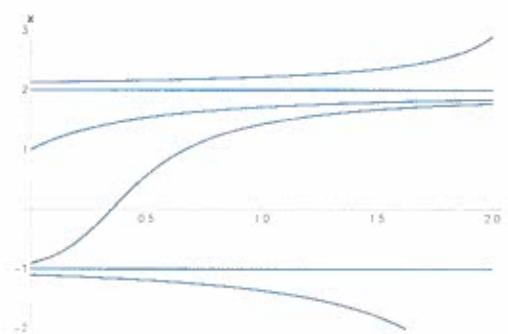
Problem 2. For the following equations sketch the vector fields on the real line, if possible find all fixed points, classify their stability and sketch the graph of $x(t)$ for different initial conditions. You must include enough sketches of $x(t)$ to illustrate all qualitatively different solution solution curves.

(a) $\dot{x} = 1 - x^{14}$.

(b) $\dot{x} = e^{-x}\sin(x)$.

(c) $\dot{x} = e^{x} - \cos(x)$ (You won't be able to find the fixed points explicitly, but you can still determine the qualitative behavior).

Problem 3. The curves $x(t)$ illustrated below correspond to solution curves for the differential equation $\dot{x} = f(x)$.



(a) Sketch a one dimensional phase portrait that is consistent with this figure.

(b) Sketch a graph of $f(x)$ that is consistent with this figure.

(c) Give a formula for $f(x)$ that is consistent with this figure.

Problem 4. For each of parts (a)-(e), find an equation $\dot{x} = f(x)$ with the stated properties, or if there are no examples, explain why not.

(a) Every real number is a fixed point.

(b) Every integer is a fixed point, and there are no others.

(c) There are precisely three fixed points, and there are no others.

(d) There are precisely three fixed points, and all of them are stable.

(e) There are no fixed points.

(f) There are precisely 100 fixed points.

Problem 5. The velocity $v(t)$ of a skydiver falling to the ground is governed by the equation $m\dot{v} = mg - kv^2$, where m is the mass of the skydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to air resistance.

(a) Obtain the analytic solution for $v(t)$, assuming that $v(0) = 0$.

(b) Find the limit of $v(t)$ as $t \to \infty$. This limiting velocity is called the terminal velocity.

(c) Give a graphical analysis of this problem, and thereby re-derive a formula for the terminal velocity.

Problem 6. Consider the following initial value problem

$$\dot{x} = 1 + x^2 \text{ and } x(0) = x_0.$$

By explicitly solving this differential equation, show that there exists a finite time t_f such that

$$\lim_{t \to t_f^-} x(t) = \infty.$$

This phenomenon is called finite time blow up.

Problem 7. Suppose X and Y are two species that reproduce exponentially fast: $\dot{X} = aX$ and $\dot{Y} = bY$, respectively, with initial conditions $X_0, Y_0 > 0$ and growth rates $a, b > 0$. Let $x(t) = X(t)/(X(t) + Y(t))$ denotes X's share of the total population.

(a) Show that $\dot{x} = (a - b)x(1 - x)$.

(b) Show that if $a > b$ then x is monotonically increasing and approaches 1 as $t \to \infty$. What does this result imply about the population?

(c) Show that if $a < b$ then x is monotonically decreasing and approaches 0 as $t \to \infty$. What does this result imply about the population?

(d) What happens if $a = b$?

Problems to be completed by graduate students

Problem 8. A particle travels on the half line $x \geq 0$ with a velocity given by $\dot{x} = -x^c$, where c is real and constant.

(a) Find all values of c such that the origin $x = 0$ is a stable fixed point.

(b) Now assume that c is chosen such that $x = 0$ is stable. Can the particle ever reach the origin in finite time? Specifically, how long does it take for the particle to travel from $x = 1$ to $x = 0$, as a function of c?

Problem 9. Prove that solutions to the initial value problem $\dot{x} = 1 + x^{10}$ blow up in finite time starting from any initial condition. **Hint:** Don't try to find an exact solution, instead compare the solutions to those of $\dot{x} = 1 + x^2$.

Problem 10. Show that the initial value problem $\dot{x} = x^{1/3}$, $x(0) = 0$, has an infinite number of solutions. **Hint:** Construct a solution that stays at $x = 0$ until some arbitrary time t_0, after which it takes off.

Homework #1

#1

Consider the system $\dot{x} = \sin(x)$.

(a) Find all fixed points of the flow.

(b) At which points x does the flow have the greatest velocity to the right.

(c) Find the flows accleration as a function of x.

(d) Find the points where the flow has maximum positive accleration.

Solution:

(a) The fixed points are given by $x^* = n\pi$, where $n \in \mathbb{Z}$.

(b) Since $\sin(x)$ is maximized at $x = \pi/2 + 2n\pi$, where $n \in \mathbb{Z}$, these are precisely the points where $\dot{x}$ is maximized.

(c) Differentiating we have

$$\ddot{x} = \frac{d}{dt}\sin(x) = \cos(x)\cdot\dot{x} = \cos(x)\cdot\sin(x) = \frac{1}{2}\sin(2x).$$

(d) The points where the flow has maximum positive acceleration are thus $x^* = \pi/4 + n\pi$, where $n \in \mathbb{Z}$.

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#2

For the following equations, sketch the vector fields on the real line, if possible find all fixed points, classify their stability and sketch the solution curves $x(t)$

(a) $\dot{x} = 1 - x^{14}$,

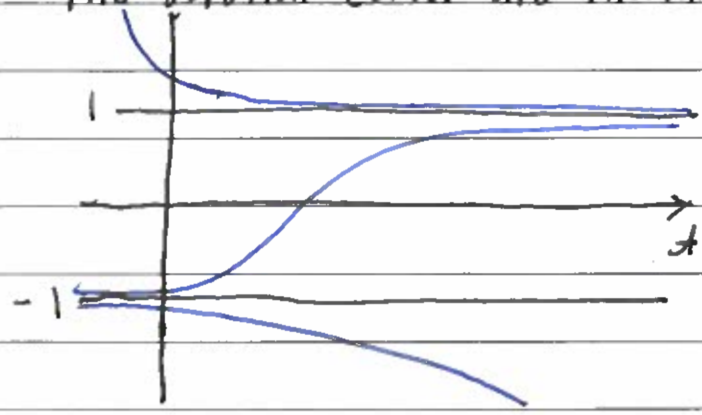
(b) $\dot{x} = e^{-x}\sin(x)$,

(c) $\dot{x} = e^{x} - \cos(x)$,

Solution:

(a) The fixed points are $x^* = \pm 1$. Since $\lim_{x\to\infty} 1 - x^{14} = -\infty$, it follows that $x^* = 1$ is stable and $x^* = -1$ is unstable. The phase portrait is

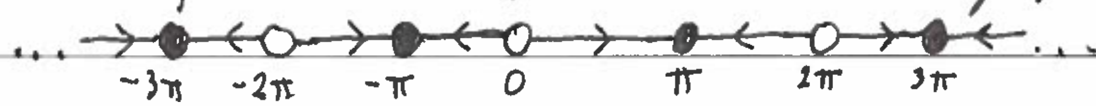
The solution curves are therefore:

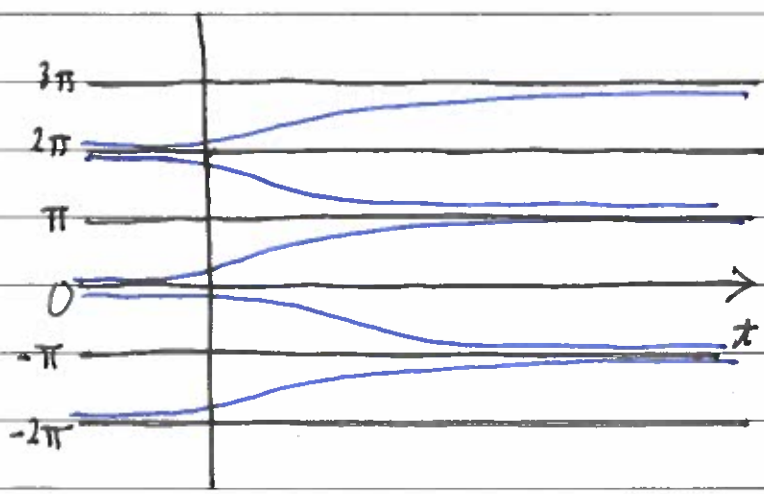


(b). The fixed points are $x^* = n\pi$, where $n \in \mathbb{Z}$. Letting $f(x) = e^{-x}\sin(x)$ we have that

$$f'(x) = -e^{-x}\sin(x) + e^{-x}\cos(x),$$
$$\Rightarrow f'(0) = 1,$$

and thus $x^* = 0$ is unstable. Therefore, since the stability of fixed points alternates, we have the following phase portrait





(c) The only fixed point that can be explicitely calculated is $x^*=0$. The other fixed points are all negative and are approximately given by $x^* \approx -\pi/2 - k\pi$, where $k \in \mathbb{N}$. Since $\lim_{x\to\infty} e^x - \cos(x) = \infty$ it follows that $x^*=0$ is unstable. The phase portrait is therefore

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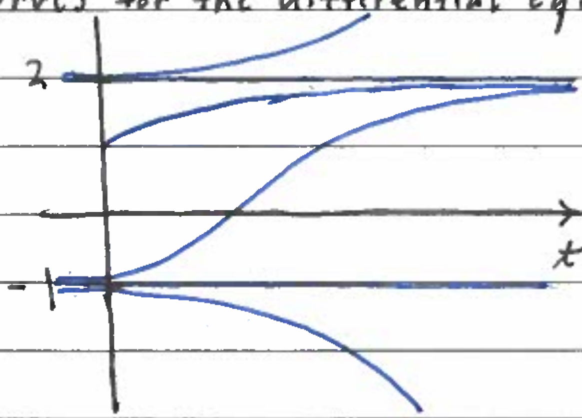
#3

The curves $x(t)$ illustrated below correspond to solution curves for the differential equation $\dot{x} = f(x)$



(a) Sketch a one dimensional phase portrait that is consistent with this figure.

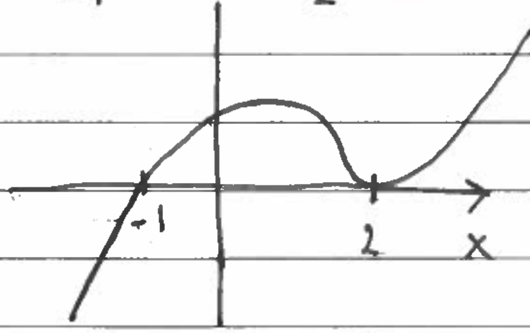
(b) Sketch a graph of $f(x)$ that is consistent with this figure.

(c) Give a formula for $f(x)$ that is consistent with this figure.

Solution:

(a) 

(b)



(c) $f(x) = (x+1)(x-2)^2$

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#5

The velocity $v(t)$ of a skydiver falling to the ground is governed by the equation $m\dot{v} = mg - kv^2$, where m is the mass of the skydiver, g is the accleration due to gravity, and $k > 0$ is a constant related to air resistance.

(a) Obtain the analytic solution assuming $v(0) = 0$.

(b) Find the limit as $t \to \infty$.

(c) Give a graphical analysis of this problem.

Solution:

(a) Seperating variables we have that:

$$\int_0^v \frac{m}{mg - kw^2}\,dw = \int_0^t ds$$

$$\Rightarrow \frac{1}{g}\int_0^v \frac{1}{(1-\sqrt{k/mg}\,w)(1+\sqrt{k/mg}\,w)}\,dw = t$$

$$\Rightarrow \frac{1}{2g}\int_0^v \frac{1}{1-\sqrt{k/mg}\,w}\,dw + \frac{1}{2g}\int_0^v \frac{1}{1+\sqrt{k/mg}\,w}\,dw = t$$

$$\Rightarrow \frac{-1}{2g}\ln(1-\sqrt{k/mg}\,v) + \frac{1}{2g}\ln(1+\sqrt{k/mg}\,v) = t \cdot \sqrt{k/mg}$$

$$\Rightarrow \ln\left(\frac{1+\sqrt{k/mg}\,v}{1-\sqrt{k/mg}\,v}\right) = 2\sqrt{\frac{kg}{m}}\,t$$

$$\Rightarrow 1+\sqrt{k/mg}\,v = (1-\sqrt{k/mg}\,v)\exp(2\sqrt{kg/m}\,t)$$

$$\Rightarrow \sqrt{k/mg}(1+e^{2gt})v = \exp(2\sqrt{kg/m}\,t)$$

$$\Rightarrow \boxed{v = \sqrt{\frac{mg}{k}}\,\frac{e^{\lambda t}-1}{1+e^{\lambda t}}}, \quad \boxed{\lambda = 2\sqrt{\frac{kg}{m}}}$$

(b) Taking the limit, we have $\lim_{t\to\infty} v(t) = \sqrt{mg/k}$.

(c) For this system the stable fixed point is $v^* = \sqrt{mg/k}$ which matches the limit we computed.

#4.

For each of parts (a)-(e), find an equation $\dot{x}=f(x)$ with the stated properties, or if there are no examples, explain why not.

(a) Every real number is a fixed point.

(b) Every integer is a fixed point and there are no others.

(c) There are precisely three fixed points and no others.

(d) There are precisely three fixed points, and all of them are stable.

(e) There are precisely 100 fixed points.

Solution:

(a) $\dot{x}=0$

(b) $\dot{x}=\sin(\pi x)$

(c) $\dot{x}=x(x-1)(x-2)$

(d) Not possible for smooth f.

(e) $\dot{x}=x(x-1)(x-2)\cdots(x-99)$.

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#8

A particle travels on the half line $x \geq 0$ with velocity $\dot{x}=-x^c$, where c is real and constant.

(a) Find all values of c such that $x=0$ is a stable fixed point.

(b) Can the particle ever reach the origin in finite time.

Solution:

(a) For $c>0$, the origin is a stable fixed point.

(b) For $c>0$, we can explicitly solve this equation:

$$\dot{x} = -x^c$$

$$\Rightarrow \int_{x_0}^{x} -y^{-c}\,dy = t$$

$$\Rightarrow \begin{cases} \dfrac{-1}{1-c}(x^{1-c} - x_0^{1-c}) = t, & c \geq 0, c \neq 1 \\ x(t) = x_0 e^{-t}, & c = 1 \end{cases}$$

$$\Rightarrow x(t) = \begin{cases} (-(1-c)t + x_0^{1-c})^{1/1-c}, & c > 0, c \neq 1 \\ x_0 e^{-t}, & c = 1 \end{cases}$$

Now, if $c \neq 1$ and we try to solve:

$$(-(1-c)t + x_0^{1-c})^{1/1-c} = 0$$

$$\Rightarrow t = \frac{x_0^{1-c}}{1-c}$$

which yields a positive and finite solution if $c < 1$. This result matches what we expect from the existence and uniqueness theorem, i.e., unique solutions exist if $c \geq 1$. ∎