Homework #2
MTH 351/651: Fall 2025
Instructor: John Gemmer

Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a "rough draft"; your submission with well-organized calculations and relevant explanations should be thought of as your "final draft".

Problems to be completed by all students

Problem 1. The following system of differential equations models the spread of infectious disease in which the recovery time decreases with the number of infected individuals:

$$\begin{aligned} \dot{S} &= -\beta I \frac{S}{N} + \frac{\alpha I}{\kappa + I}, \\ \dot{I} &= \beta I \frac{S}{N} - \frac{\alpha I}{\kappa + I}, \end{aligned} \tag{1}$$

where β, α, and κ are positive constants. Here I denotes the. number of infected individuals, S denotes the number of susceptible individuals and $N = S + I$ denotes the total population size.

(a) Show that the total population is constant in time.

(b) Using the fact that N is constant, reduce Equation (??) into one differential equation for the number of infected individuals:

$$\dot{I} = \beta I \frac{N - I}{N} - \frac{\alpha I}{\kappa + I}. \tag{2}$$

(c) By changing variables to $x = I/N$ and $\tau = \beta t$, show that Equation (??) is equivalent to the following differential equation

$$\frac{dx}{d\tau} = x(1 - x) - \frac{ax}{1 + bx}, \tag{3}$$

for appropriately chosen positive constants a and b.

(d) Assuming $a = 1$, find all possible fixed points for Equation (??), analyze their stability, and plot all qualitatively different phase portraits that can occur as b is varied. Specifically, phase portraits are qualitatively different if they have a different number of fixed points and/or the stability of the fixed points changes. Interpret in practical terms what the different phase portraits tells you about the spread of the disease in different parameter regimes.

(e) Assuming $b = 1$, find all possible fixed points for Equation (??), analyze their stability, and plot all qualitatively different phase portraits that can occur as a is varied. Specifically, phase portraits are qualitatively different if they have a different number of fixed points and/or the stability of the fixed points changes. Interpret in practical terms what the different phase portraits tells you about the spread of the disease in different parameter regimes.

Problem 2. Consider the following dynamical system $\dot{x} = ax - x^3$ where a is a real number that can be positive, negative, or zero.

(a) For all three cases find the fixed points, classify their stability, and sketch the graph of $x(t)$ for different initial conditions.

(b) For all three cases, calculate and plot the potential function $V(x)$.

Problem 3. Recall that the improved Euler method is given by

$$\tilde{x}_{n+1} = x_n + f(x_n)\Delta t,$$
$$x_{n+1} = x_n + \frac{1}{2}\left[f(x_n) + f(\tilde{x}_{n+1})\right].$$

Use Taylor series to show that the local error of the improved Euler method is $O(\Delta t^3)$.

Problems to be completed by graduate students

Problem 4. Consider the differential equation

$$\dot{x} = f(x),$$
$$x(0) = x_0,$$

where f is a smooth function with isolated zeros. Prove that if x_0 is not a fixed point for this system then there does not exist times t_1 and t_2 in which $x(t_1) = x(t_2)$, i.e., the solution cannot oscillate.

Homework #2

#1

$$\dot{S} = -\beta I \, S/N + \frac{\alpha I}{K+I}$$
$$\dot{I} = \beta I \, S/N - \frac{\alpha I}{K+I}.$$

Solution:

(a) Differentiating we have that

$$\dot{N} = \dot{S} + \dot{I} = 0$$

and thus N is constant

(b) Solving for S we have that $S = N - I$ and thus

$$\dot{I} = \frac{\beta I}{N}(N-I) - \frac{\alpha I}{K+I}.$$

(c) Letting $x = I/N$ and $\tau = \beta t$ we have that

$$\frac{dI}{dt} = \frac{dI}{d\tau}\frac{d\tau}{dt} = \beta \frac{dI}{d\tau} = \beta N \frac{dx}{d\tau}$$

Therefore,

$$\beta N \frac{dx}{d\tau} = \frac{\beta I}{N}(N-I) - \frac{\alpha I}{K+I} = \beta N x(1-x) - \frac{\alpha N x}{K(1+N/K\, x)}$$

$$\Rightarrow \frac{dx}{d\tau} = x(1-x) - \frac{\alpha x}{\beta K(1+N/K\, x)} = x(1-x) - \frac{ax}{1+bx},$$

where $a = \alpha/\beta K$ and $b = N/K$.

(d) Setting $a=1$, we have that

$$\frac{dx}{d\tau} = x(1-x) - \frac{x}{1+bx}$$

The fixed points satisfy

$$x^* = 0, \quad (1-x^*) - \frac{1}{1+bx^*} = 0$$

$$\Rightarrow x^* = 0, \quad \cancel{1} + bx^* - x^* - bx^{*2} - \cancel{1} = 0$$

$$\Rightarrow x^* = 0, \quad x^* = (b-1)/b.$$

We have two cases:

Case #1 ($b<1$):

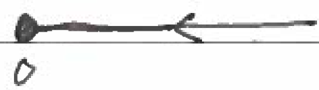
In this there is one nonnegative fixed point. Letting

$$f(x) = x(1-x) - \frac{x}{1+bx} = \frac{x^2}{1+bx}(b-1-bx)$$

it follows that

$$\lim_{x\to 0^+} \frac{1}{x^2} f(x) = b-1 < 0$$

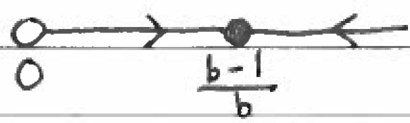
and thus $x^* = 0$ is stable and we have the phase portrait on the half line $I \geq 0$:



In this case the recovery rate is large enough that the disease dies out.

Case #2 ($b>1$):

Since $\lim_{x\to 0^+} \frac{1}{x^2} f(x) > 0$ and $\lim_{x\to\infty} f(x) = -\infty$ we have the following phase portrait



In this case the recovery rate is small enough so that the number of infected becomes endemic at the population level $x^* = \frac{b-1}{b} \Rightarrow I^* = \frac{N}{b}(b-1$

(e) Setting $b=1$, we have

$$\frac{dx}{d\tau} = x(1-x) - \frac{ax}{1+x}$$

The fixed points satisfy

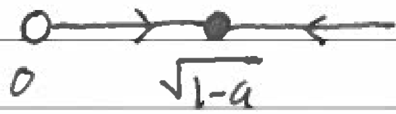
$$x^* = 0, \quad 1 - x^2 - a = 0$$

$$\Rightarrow x^* = 0, \quad x^* = \pm\sqrt{1-a}$$

Again, we have two cases:

Case #1 ($a<1$):

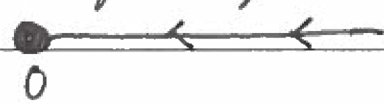
Letting $f(x) = x(1-x) - ax/1+x$ it follows that $\lim_{x\to\infty} f(x) = -\infty$. Since there are no repeated roots we have the following phase portrait



Therefore, the recovery rate is small enough that the disease becomes endemic.

Case #2 ($a>1$):

Again, since $\lim_{x\to\infty} f(x) = -\infty$, it follows that the phase portrait is given by



Therefore, the recovery rate is large enough that the disease dies out. ∎

#2

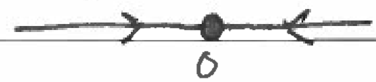
Consider the dynamical system $\dot{x} = ax - x^3$, where $a \in \mathbb{R}$.

Solution:

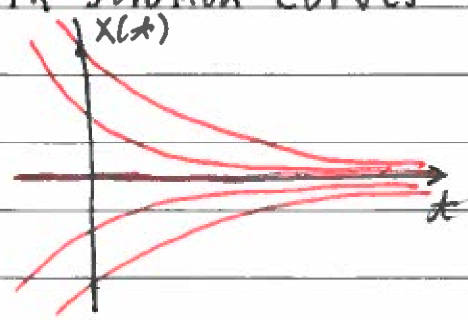
In all cases, the fixed points are $x=0$ and $x^* = \pm\sqrt{a}$.

(a) Case 1 ($a<0$):

The only fixed point is $x^*=0$, since $\lim_{x\to\infty} ax - x^3 = -\infty$ it follows that the phase portrait is

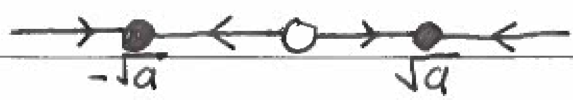


With solution curves

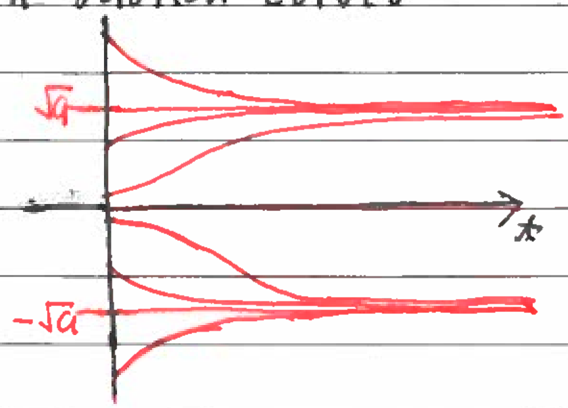


Case 2 ($a>0$):

Again since $\lim_{x\to\infty} ax - x^3 = -\infty$ it follows that the phase portrait is



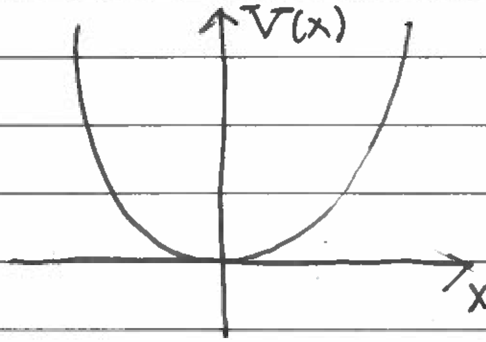
With solution curves



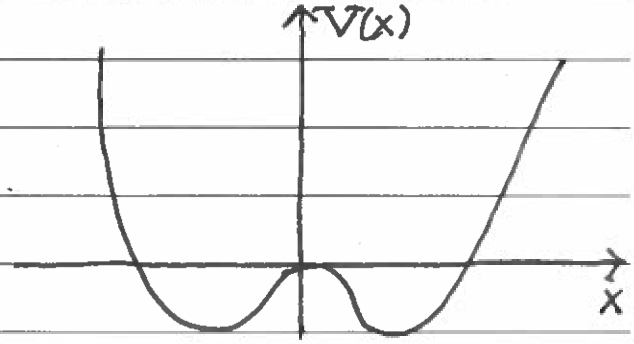
(b) The potential is given by

$$V(x) = \frac{1}{4}x^4 - \frac{1}{2}ax^2.$$

Case 1 $(a<0)$:



Case 2 $(a>0)$:



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#3

$$\dot{x} = f(x)$$

$$\tilde{x}_{n+1} = x_n + f(x_n)\Delta t$$

$$x_{n+1} = x_n + \frac{1}{2}\left(f(x_n) + f(\tilde{x}_{n+1})\right)\Delta t$$

Solution:

We have that

$$x_1 = x_0 + \frac{1}{2}\left(f(x_0) + f(x_0 + f(x_0)\Delta t\right)\Delta t$$

Taylor expanding in Δt we have that

$$x_1 = x_0 + \frac{1}{2}\left(f(x_0) + f(x_0) + f'(x_0)f(x_0)\Delta t\right)\Delta t + \mathcal{O}(\Delta t^3)$$

$$= x_0 + f(x_0)\Delta t + \frac{1}{2}f(x_0)f'(x_0)\Delta t^2 + \mathcal{O}(\Delta t^3)$$

We can also Taylor expand the exact solution about Δt:

$$x(\Delta t) = x(0) + \dot{x}(0)\Delta t + \frac{1}{2}\ddot{x}(0)\Delta t^2 + \mathcal{O}(\Delta t^3)$$

$$= x_0 + f(x_0)\Delta t + \frac{1}{2}f(x_0)f'(x_0)\Delta t^2 + \mathcal{O}(\Delta t^3)$$

Therefore, the local error is given by

$$|x(\Delta t) - x_1| = \mathcal{O}(\Delta t^3).$$

Consequently, the global error is $\mathcal{O}(\Delta t^2)$.

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#4

~~proof~~:

If we let $V(x) = -\int_0^x f(y)\,dy$, it follows that for a solution $x(t)$ of $\dot{x} = f(x)$ that

$$\frac{d}{dt}V(x(t)) = \frac{dV}{dx}\Big|_{x(t)} \cdot \dot{x} = -f(x(t))^2 \le 0.$$

If there exists times t_1, t_2 in which $x(t_1) = x(t_2)$ it follows that $V(x(t_1)) = V(x(t_2))$. However, since $V(x(t))$ is decreasing it follows that $V(x(t))$ is constant on the interval $[t_1, t_2]$ which is only possible if

$$\frac{d}{dt}V(x(t)) = -f(x(t)) = 0$$

on $[t_1, t_2]$ which is only possible if $x(t)$ is a fixed point.