Homework #3
MTH 351/651: Fall 2025
Instructor: John Gemmer

Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a "rough draft"; your submission with well-organized calculations and relevant explanations should be thought of as your "final draft".

Problems to be completed by all students

3 **Problem 1.** Consider the following model for the population of infected individuals during an epidemic:

$$\dot{I} = \alpha I(N - I) - \beta I - f(I)I,$$

where

$$f(I) = \frac{\gamma}{A + I},$$

and $\alpha, \beta, N, \gamma, A > 0$ are all parameters. Note, we are implicitly assuming $I \geq 0$.

(a) Calculate $f(0)$, $\lim_{I \to \infty} f(I)$, $f'(I)$, and $f''(I)$. Use this information to sketch a graph of $f(I)$ on the positive real axis.

(b) Give a practical interpretation of the function $f(I)$. That is, thinking of $f(I)$ as an additional recovery rate, what could this function model in the real world?

(c) Determine the dimensions of the parameters $\alpha, \beta, N, \gamma, A$.

(d) Show that there exists a dimensionless change of variables in I and t such that this system is equivalent to the following dimensionless system

$$\frac{dx}{d\tau} = ax(1 - x) - x - \frac{bx}{c + x},$$

for some appropriate constants a, b, c.

(e) For this dimensionless system, calculate the fixed points for this system and analyze their stability.

(f) Show that there exists a different dimensionless change of variables in I and t such that this system is equivalent to the following dimensionless system

$$\frac{dx}{d\tau} = ax(b - x) - cx - \frac{x}{1 + x},$$

for some other appropriate constants a, b, c.

2 **Problem 2.** In this problem you will be considering three simple models of fish in a fishery. Let $N(t)$ be the population of fish at time t. In the the absence of fishing, the population is assumed to grow logistically. Three possible models are as follows:

$$\textbf{Model \#1: } \dot{N} = rN\left(1 - \frac{N}{\kappa}\right) - H_1,$$

$$\textbf{Model \#2: } \dot{N} = rN\left(1 - \frac{N}{\kappa}\right) - H_2 N,$$

$$\textbf{Model \#2: } \dot{N} = rN\left(1 - \frac{N}{\kappa}\right) - H_3 \frac{N}{A+N},$$

where H_1, H_2, and H_3, and A are positive constants.

(a) For each model, give a biological interpretation of the fishing term. How do they differ? What is the meaning of the constants H_1, H_2, H_3, and A?

(b) Why is model #1 not biologically realistic?

(c) Which of models #2 or #3 do you think is best and why?

Problem 3. Modify the Matlab code for Euler's method to create your own code that implements the modified Euler method. Use your code to plot all possible qualitatively different solution curves for Models #1, #2, #3 for the parameter values $r = 1, \kappa = 5$, $H_1 = 1$, $H_2 = 2$, $H_3 = 3$, and $A = 1$ over the interval of time $[0, 11]$. You can just submit printed out versions of your plots.

2 **Problem 4.** One possible way that populations can be modeled is not by modeling the number of individuals but by the fraction of patches of land that a species can occupy. For example,

$$\dot{P} = cP(h - P) - \mu P,$$

where $P(t)$ denotes the fraction of occupied patches. The number h denotes the fraction of patches that are actually habitable for the population and, hence, $h - P$ is the number of empty but habitable patches. Note, $0 \leq P \leq h \leq 1$.

(a) Determine the dimensions of the constants c, h, μ and provide practical interpretations of c and μ.

(b) Find the equilibrium for this system and analyze their stability.

(c) For fixed values of c and μ, sketch a bifurcation diagram with h as the parameter.

(d) Do all the habitable patches have to be destroyed before the population dies out?

3 **Problem 5.** For each of the following problems sketch all qualitatively different phase portraits that occur as r is varied. In each problem sketch a bifurcation diagram of fixed points x versus r and determine what type of bifurcation occurs.

(a) $\dot{x} = 1 + rx + x^2$

(b) $\dot{x} = rx + x^2$

(c) $\dot{x} = r - \frac{1}{2}\left(e^x - e^{-x}\right)$

(d) $\dot{x} = x - rx(1-x)$

(e) $\dot{x} = x + \dfrac{rx}{1+x^2}$

(f) $\dot{x} = r - 3x^3$

(g) $\dot{x} = rx - \dfrac{x}{1+x^2}$

(h) $\dot{x} = rx + \dfrac{x^3}{1+x^2}$

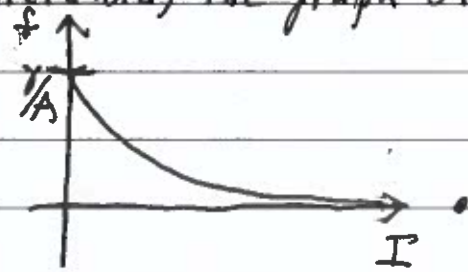
Homework #3

#1

(a) If $f(I) = \gamma/(A+I)$ then,

- $f(0) = \gamma/A$
- $f'(0) = -\gamma/(A+I)^2$
- $f''(0) = 2\gamma/(A+I)^3$

Therefore, the graph of f is given by



(b) $f(I)$ represents a recovery rate that declines with the number of infected. This could model the effect of medical resources as they get overwhelmed by the spread of the disease.

(c) $[\alpha] = \frac{1}{\text{pop}\cdot\text{time}}$

$[N] = \text{population}$

$[\beta] = \frac{1}{\text{time}}$

$[\gamma] = \frac{\text{pop}}{\text{time}}$

$[A] = \text{pop}.$

(d) Let $x = I/N$ and $\tau = \beta t$. Therefore,

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \beta\frac{d}{d\tau}.$$

Consequently,

$$\dot{I} = \frac{d}{dt}(Nx) = N\beta\frac{dx}{d\tau}$$

$$N\beta\frac{dx}{d\tau} = \alpha N^2 x(1-x) - \beta N x - \frac{\gamma N x}{A + Nx}$$

$$\Rightarrow \frac{dx}{d\tau} = ax(1-x) - x - \frac{bx}{c+x},$$

Where $a = \alpha N/\beta$, $b = \gamma/\beta N$, $c = A/N$.

(e) The fixed points satisfy

$$x=0 \quad \text{and} \quad a(1-x)(c+x)-(c+x)-b=0$$

$$\Rightarrow x=0 \quad \text{and} \quad a(c+(1-c)x-x^2)-c-x-b=0$$

$$\Rightarrow x=0 \quad \text{and} \quad ax^2+(ca-a+1)x+b+c-ac=0$$

$$\Rightarrow x=0 \quad \text{and} \quad x=\frac{a-ca-1\pm\sqrt{(ca-a+1)^2-4a(b+c-ac)}}{2a}=r_{1,2}.$$

We also have that if we let $g(x)=ax(1-x)-x-\frac{bx}{c+x}$ and thus

$$g'(0)=a-1-b/c.$$
$$=\frac{ac-c-b}{c}.$$

Therefore, 0 is a stable fixed point if $ac-c-b<0$ which implies

$$a<b/1+c$$

Moreover,

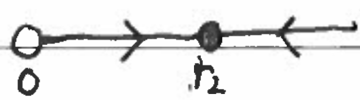
$$\lim_{x\to\infty} g(x)=-\infty.$$

Consequently, the rightmost fixed point is stable. The possible cases are therefore

Case #1 ($a<b/1+c$, $(ca-a+1)^2-4a(b+c-ac)<0$)

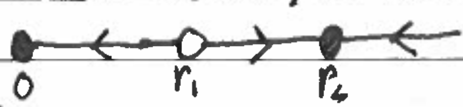
Disease goes extinct.

Case #2 ($a>b/1+c$, $(ca-a+1)^2-4a(b+c-ac)>0$)



The disease goes to an endemic state at r_2

Case #3 ($a<b/1+c$, $(ca-a+1)^2-4a(b+c-ac)>0$)



The disease can go extinct or become endemic.

(f) Letting $x = I/A$ and $\tau = \gamma/A\, t$. Therefore,

$$A\frac{\gamma\, dx}{A\, d\tau} = \alpha A x(N - Ax) - \beta A x - \frac{\gamma A x}{A + Ax}$$

$$\Rightarrow \frac{dx}{d\tau} = ax(b-x) - cx - \frac{x}{1+x},$$

where

$a = \alpha A^2/\gamma$, $b = N/A$, and $c = \beta A/\gamma$.

■

#2

(a) For model #1, H_1 is a constant removal of fish regardless of population size. For model #2, H_2 is the fishing rate assuming the amount removed is just proportional to population size. For model #3, the rate of removal is population dependent and declines depending on the number of fish. If $N = A$ we have that the rate of fishing is reduced by a factor of $1/2$ from its maximum rate. Consequently, A measures a critical threshold for the population in terms of a removal rate. That is if $N > A$ the rate of fishing is small while if $N < A$ the rate of fishing is large.

(b) Model #1 is not realistic since fish are removed even if $N = 0$.

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#4.

(a) $[c] = \frac{1}{time}$

$[h] =$ dimensionless

$[\nu] = \frac{1}{time}$

The constant c is the ideal rate that patches become occupied while ν is the rate patches become uninhabtible.

(b) Setting $\dot{P} = 0$ we have

$$cP(h-P) - \nu P = 0$$

$$\Rightarrow P(ch - cP - \nu) = 0$$

$$\Rightarrow P^* = 0,\ P^* = \frac{ch-\nu}{c}.$$

Letting $f(P) = cP(h-P) - \nu P$ we have that $f'(0) = ch - \nu$ and thus, since there are only two fixed points, we have two cases

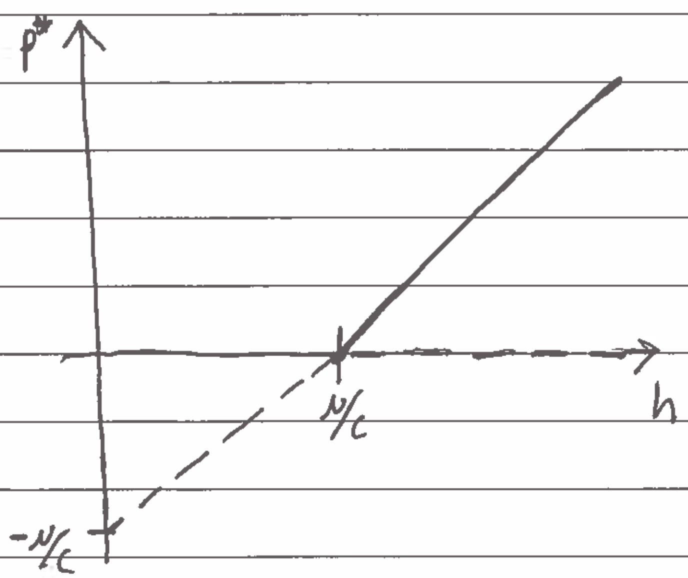
Case #1 ($h < \nu/c$):

In this case $P^* = 0$ is stable and $P^* = (ch-\nu)/c$ is unstable.

Case #2 ($h > \nu/c$):

In this case $P^* = 0$ is unstable and $P^* = (ch-\nu)/c$ is stable.

(c)



(d) No, if $h < {}^{N}/_{c}$, the population will still die out.

#5

(e) $\dot{x} = x + \dfrac{rx}{1+x^2} = f(x)$.

The fixed points satisfy

$x^* = 0,\quad 1 + x^2 + r = 0$

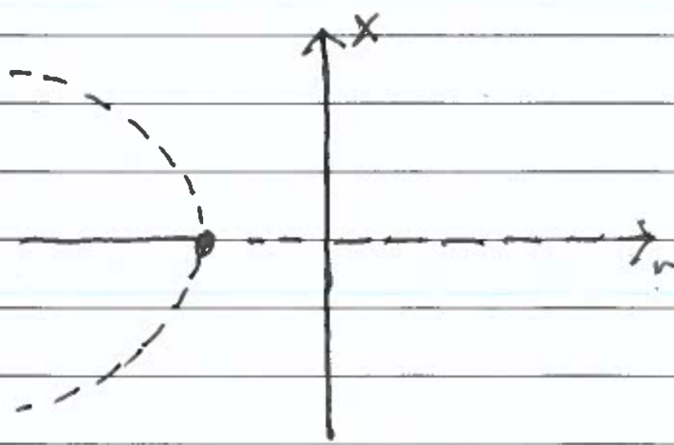
$\Rightarrow x^* = 0,\quad x^* = \pm\sqrt{-r-1}$

Since $\lim_{x\to\infty} f(x) = \infty$, we have the following cases:

Case #1: $r \le -1$

Case #2: $r > -1$

Therefore, we have a subcritical pitchfork bifurcation with bifurcation diagram:



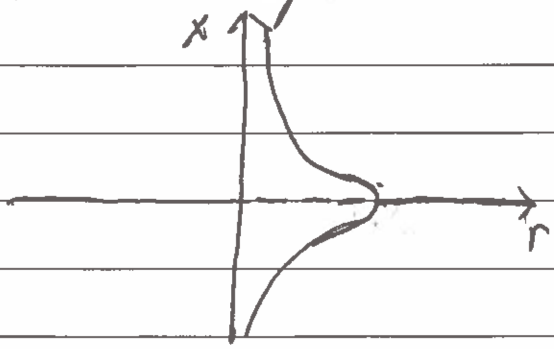
(g) $\dot{x} = rx - \dfrac{x}{1+x^2}$

The fixed points satisfy

$$x = 0, \quad r(1+x^2) - 1 = 0$$

$$\Rightarrow x = 0, \quad r = \frac{1}{1+x^2} \quad \text{or} \quad x = \pm\sqrt{\frac{1}{r} - 1}$$

Since $\lim\limits_{x\to\infty} rx - \frac{x}{1+x^2} = \text{sign}(r)\infty$, Therefore, we have the following bifurcation diagram



(h) $\dot{x} = rx + \dfrac{x^3}{1+x^2}$

The fixed points satisfy

$$x = 0, \quad r(1+x^2) + x^2 = 0$$

$$\Rightarrow x = 0, \quad r = -\frac{x^2}{1+x^2}$$

We also have that $\lim\limits_{x\to\infty} \dot{x} = \text{sign}(r+1)\infty$. If we first plot $r(x)$, rotate, and then apply stability we have: