

Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a “rough draft”; your submission with well-organized calculations and relevant explanations should be thought of as your “final draft”.

Problems to be completed by all students

Problem 1. The following system of differential equations models the salinity of the oceans in the northern (S_1) and southern (S_2) latitudes:

$$\begin{aligned}\dot{S}_1 &= -H + |q|(S_2 - S_1), \\ \dot{S}_2 &= H + |q|(S_1 - S_2),\end{aligned}$$

where $q = k(2\alpha T^* - \beta(S_2 - S_1))$, T^* denotes the temperature of the northern latitude oceans, and $k, \alpha, \beta, H > 0$ are all parameters. Note, T^* can be either positive or negative.

(a) Letting $\Delta S = S_2 - S_1$, show that

$$\frac{d\Delta S}{dt} = 2H - 2k|2\alpha T^* - \beta\Delta S|\Delta S.$$

(b) Show by making the following change of variables

$$x = \frac{\beta}{2\alpha T^*}\Delta S, \quad \tau = 4\alpha k|T^*|t, \quad \lambda = \frac{\beta H}{4\alpha^2 k T^* |T^*|},$$

that the system can be nondimensionalized into the form:

$$\frac{dx}{d\tau} = \lambda - |1 - x|x.$$

- (c) Determine the fixed points for this system and analyze their stability as a function of λ .
- (d) Sketch a bifurcation diagram for this problem and explicitly calculate any bifurcation points.
- (e) What happens to the salinity of the oceans as T^* increases or decreases?

Problem 2. For each of the following vector fields, find and classify all the fixed points and sketch the phase portrait on the circle:

- (a) $\dot{\theta} = 1 + 2\cos(\theta)$
- (b) $\dot{\theta} = \sin(2\theta)$
- (c) $\dot{\theta} = \sin(\theta) + \cos(\theta)$
- (d) $\dot{\theta} = 3 + \cos(2\theta)$

Problem 3. Let θ_1 and θ_2 denote the angular positions of the hour and minute hands of a clock.

- (a) If time is measured in seconds, write down differential equations satisfied by t_1 and t_2 .
- (b) If at 12:00 the hour and minute hands of a clock are perfectly aligned, when is the next time they will be aligned?

Problem 4. For each of the following dynamical systems on the unit circle, draw the bifurcations diagrams, classify any bifurcations that occur as μ varies, and find all the bifurcation values of μ .

- (a) $\dot{\theta} = \mu \sin(\theta) - \sin(2\theta)$
- (b) $\dot{\theta} = \frac{\sin(\theta)}{\mu + \cos(\theta)}$
- (c) $\dot{\theta} = \mu + \cos(\theta) + \cos(2\theta)$
- (d) $\dot{\theta} = \mu + \sin(\theta) + \cos(2\theta)$.