Homework #4
MTH 351/651: Fall 2025
Instructor: John Gemmer

Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a "rough draft"; your submission with well-organized calculations and relevant explanations should be thought of as your "final draft".

Problems to be completed by all students

3 **Problem 1.** The following system of differential equations models the salinity of the oceans in the northern (S_1) and southern (S_2) latitudes:

$$\dot{S}_1 = -H + |q|(S_2 - S_1),$$
$$\dot{S}_2 = H + |q|(S_1 - S_2);$$

where $q = k(2\alpha T^* - \beta(S_2 - S_1)$, T^* denotes the temperature of the northern latitude oceans, and $k, \alpha, \beta, H > 0$ are all parameters. Note, T^* can be either positive or negative.

(a) Letting $\Delta S = S_2 - S_1$, show that

$$\frac{d\Delta S}{dt} = 2H - 2k|2\alpha T^* - \beta\Delta S|\Delta S.$$

(b) Show by making the following change of variables

$$x = \frac{\beta}{2\alpha T^*}\Delta S,\ \tau = 4\alpha k|T^*|t,\ \lambda = \frac{\beta H}{4\alpha^2 kT^*|T^*|},$$

that the system can be nondimensionalized into the form:

$$\frac{dx}{d\tau} = \lambda - |1 - x|x.$$

(c) Determine the fixed points for this system and analyze their stability as a function of λ.

(d) Sketch a bifurcation diagram for this problem and explicitly calculate any bifurcation points.

(e) What happens to the salinity of the oceans as T^* increases or decreases?

3 **Problem 2.** For each of the following vector fields, find and classify all the fixed points and sketch the phase portrait on the circle:

(a) $\dot{\theta} = 1 + 2\cos(\theta)$

(b) $\dot{\theta} = \sin(2\theta)$

(c) $\dot{\theta} = \sin(\theta) + \cos(\theta)$

(d) $\dot{\theta} = 3 + \cos(2\theta)$

2 **Problem 3.** Let θ_1 and θ_2 denote the angular positions of the hour and minute hands of a clock.

(a) If time is measured in seconds, write down differential equations satisfied by t_1 and t_2.

(b) If at 12:00 the hour and minute hands of a clock are perfectly aligned, when is the next time they will be aligned?

2 **Problem 4.** For each of the following dynamical systems on the unit circle, draw the bifurcations diagrams, classify any bifurcations that occur as μ varies, and find all the bifurcation values of μ.

1 (a) $\dot{\theta} = \mu \sin(\theta) - \sin(2\theta)$

(b) $\dot{\theta} = \frac{\sin(\theta)}{\mu + \cos(\theta)}$

1 (c) $\dot{\theta} = \mu + \cos(\theta) + \cos(2\theta)$

(d) $\dot{\theta} = \mu + \sin(\theta) + \cos(2\theta)$.

Homework #4

#1

Solution:

(a) Letting $\Delta S = S_2 - S_1$ we we have

$$\Delta\dot{S} = \dot{S}_2 - \dot{S}_1$$
$$= H + |q|(S_1 - S_2) + H - |q|(S_2 - S_1)$$
$$= 2H - 2|q|(S_2 - S_1)$$
$$= 2H - 2K|2\alpha T^* - \beta \Delta S|\Delta S.$$

(b) Letting $x = \beta/2\alpha T^* \Delta S$, $\tau = 4\alpha K|T^*|t$, $\lambda = \frac{\beta H}{4\alpha^2 K T^* |T^*|}$,

we have

$$\dot{x} = \frac{dx}{d\tau}\frac{d\tau}{dt} = 4\alpha K|T^*|\frac{dx}{d\tau}$$

However,

$$\dot{x} = \beta/2\alpha T^* \Delta\dot{S}$$
$$= \beta/2\alpha T^* (2H - 2K|2\alpha T^* - 2\alpha T^* x|\tfrac{2\alpha T^*}{\beta} x)$$
$$= \frac{\beta H}{\alpha T^*} - 4K\alpha T^*|1-x|x$$

Therefore,

$$\frac{dx}{d\tau} = \lambda - |1-x|x.$$

(c) Solving for fixed points we have the following cases

Case #1, $x<1$:

$$\lambda - (1-x)x = 0$$
$$\Rightarrow x^2 - x + \lambda = 0$$
$$\Rightarrow x = \frac{1 \pm \sqrt{1-4\lambda}}{2}$$

This only yields two solution also satisfying $x<1$ if $\lambda < 1/4$:

$$x^* = \frac{1-\sqrt{1-4\lambda}}{2}, \frac{1+\sqrt{1-4\lambda}}{2}$$

Case #2, $x>1$:

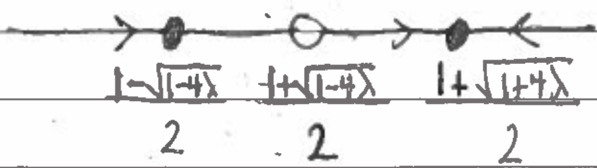
$$\lambda+(1-x)x=0$$
$$x^2-x-\lambda=0$$
$$x=\frac{1\pm\sqrt{1+4\lambda}}{2}$$

This yields one solution satisfying $x>1$ if $\lambda>-1/4$:

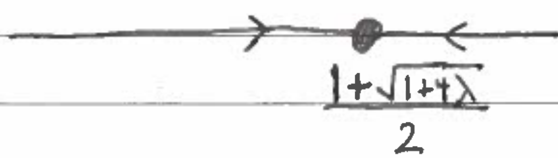
$$x^*=\frac{1+\sqrt{1+4\lambda}}{2}.$$

Since $\lim_{x\to\infty} \lambda-(1-x)x=-\infty$ and by construction $\lambda>0$ we have the following cases:

$0<\lambda<1/4$:



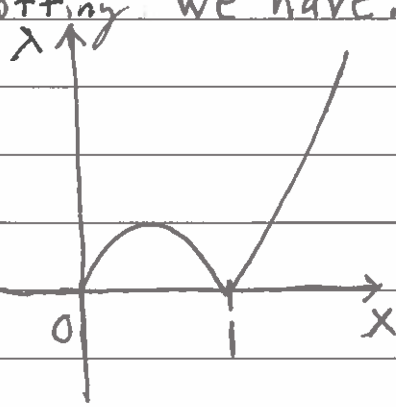
$1/4<\lambda$:



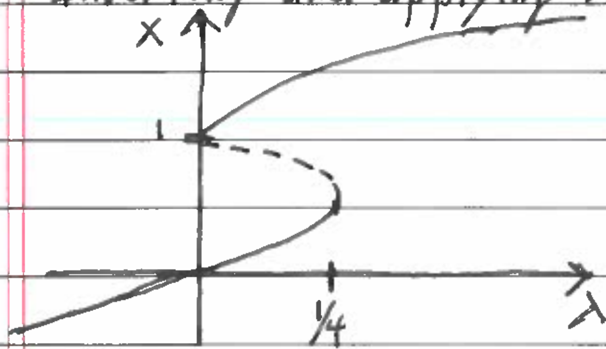
(d) The fixed point equation is

$$\lambda-|1-x|x=0$$
$$\Rightarrow \lambda=|1-x|x.$$

Plotting we have:



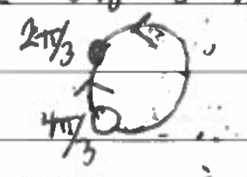
Inverting and applying stability we have:



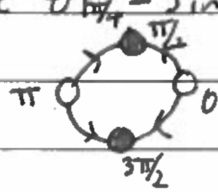
(e) As T^* increases, λ decreases. In particular, if λ crosses from $\lambda < 1/4$ to $\lambda > 1/4$ there is a sharp increase in the salinity of the ocean.

#2

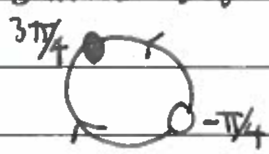
(a) The equation $\dot{\theta} = 1 + 2\cos\theta$ has fixed points at $\theta = 2\pi/3, 4\pi/3$. Since $\dot{\theta}|_0 = 3$ we have the following phase portrait:



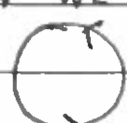
(b) The equation $\dot{\theta} = \sin(2\theta)$ has fixed points at $\theta = 0, \pi/2, \pi, 3\pi/2$. Since $\dot{\theta}|_{\pi/4} = \sin(\pi/2) = 1$ we have the following phase portrait:



(c) The equation $\dot{\theta} = \sin\theta + \cos\theta$ has fixed points at $\theta = -\pi/4, 3\pi/4$. Since $\dot{\theta}|_0 = 1$ we have the following phase portrait:



(d) The equation $\dot{\theta} = 3 + \cos(2\theta)$ has no fixed points. Since $\dot{\theta}|_0 = 4$ we have the following phase portrait:



#3

(a) $\dot{\theta}_1 = \dfrac{2\pi}{3600\cdot 12}$ and $\dot{\theta}_2 = \dfrac{2\pi}{3600}$

(b) If we let $\theta = \theta_2 - \theta_1$ we have that

$$\dot{\theta} = \frac{2\pi}{3600} - \frac{2\pi}{3600\cdot 12}$$

$$\Rightarrow \theta = \frac{2\pi}{3600}\left(\frac{11}{12}\right)t.$$

Setting $\theta = 2\pi$ we have

$$2\pi = \frac{2\pi}{3600}\left(\frac{11}{12}\right)t$$

$$\Rightarrow t = \frac{3600\cdot 12}{11} \text{ seconds}$$

$$\Rightarrow t \approx 65.5 \text{ minutes.}$$

#4

(a) $\dot{\theta} = \mu\sin\theta - 2\sin\theta\cos\theta$

$= \sin\theta(\mu - 2\cos\theta)$

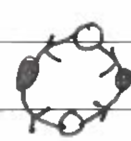
The fixed points satisfy $\theta^* = 0, \pi$ and $\cos\theta^* = \mu/2$ with the last type only existing if $-2<\mu<2$. Differentiating we have

$$\left.\frac{d\dot{\theta}}{d\theta}\right|_{\theta=0} = \mu\cos 0 - 2\cos 0 = \mu - 2$$

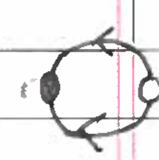
and thus $\theta^* = 0$ is stable if $\mu < 2$. We thus have the following cases

Case #1, $\mu < -2$

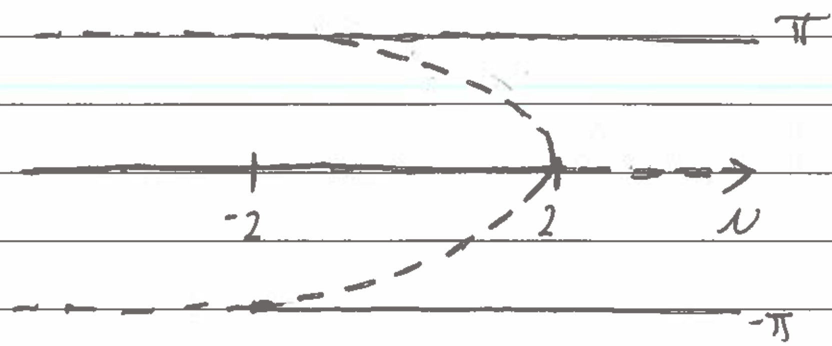
Case #2, $-2 < \mu < 2$



Case #3, $\mu > 2$



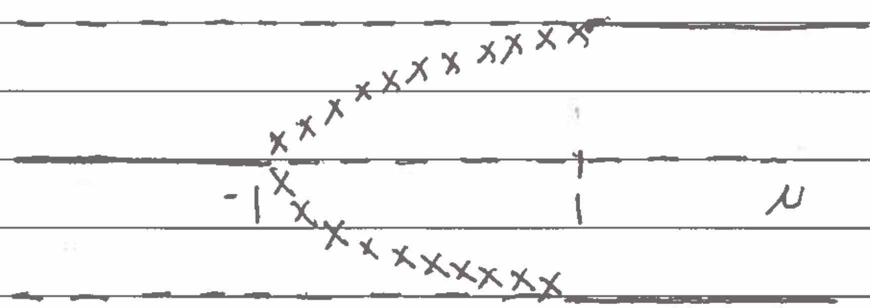
The bifurcation diagram is therefore



(b) The equation $\dot{\theta} = \sin\theta / \nu + \cos\theta$ has fixed points at $\theta^* = 0, \pi$ and asymptotes when $\cos\theta = -\nu$. Differentiating we have

$$\left.\frac{d\dot{\theta}}{d\theta}\right|_{\theta=0} = \frac{\nu+1}{\nu^2}$$

and thus $\theta^* = 0$ is stable when $\nu < -1$. The bifurcation diagram is therefore



where I used x's to denote the location of asymptotes.

(c) Simplifying we have

$$\dot{\theta} = \nu + \cos\theta + 2\cos^2\theta - 1$$

The fixed points satisfy

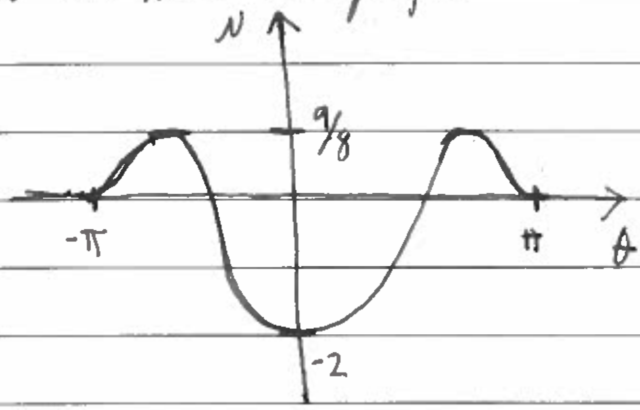
$$\nu = -2\cos^2\theta - \cos\theta + 1$$

Letting $f(\theta) = -2\cos^2\theta - \cos\theta + 1$, we have

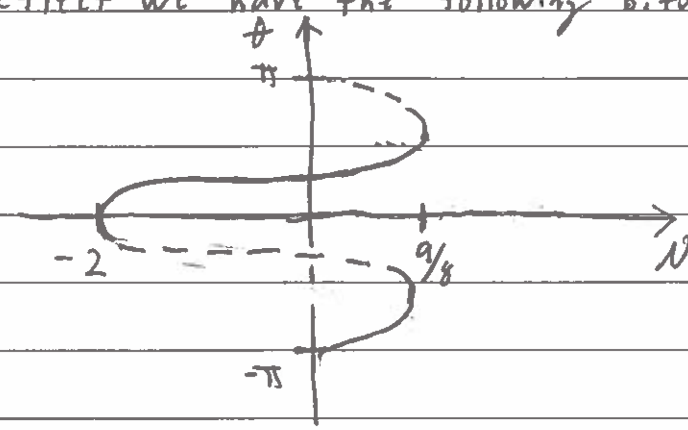
$$f'(\theta) = 4\cos\theta\sin\theta + \sin\theta$$
$$= 4\sin\theta(\cos\theta + 1/4)$$

and thus f has local maximums when $\cos\theta = -1/4$ and $\theta = 0, \pi$.

Thus we have the graph:



Now, when $\nu = 0$ and $\theta = 0$ we have $\dot{\theta}\big|_{\nu=0,\theta=0} = 1$ and thus when we reflect we have the following bifurcation diagram:



(d) Simplifying we have:

$$\dot{\theta} = \nu + \sin\theta + 1 - 2\sin^2\theta$$

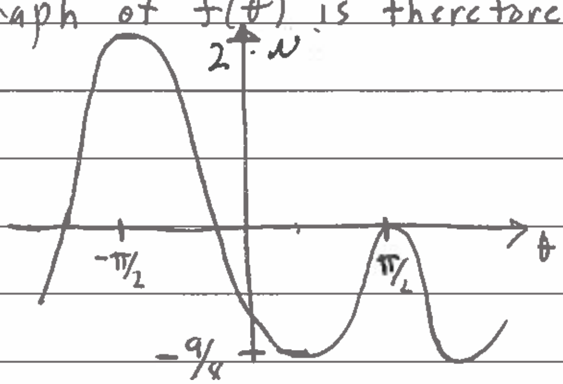
The fixed points satisfy

$$\nu = 2\sin^2\theta - \sin\theta - 1$$

Letting $f(\theta) = 2\sin^2\theta - \sin\theta - 1$ we have

$$f'(\theta) = 4\sin\theta\cos\theta - \cos\theta$$
$$= \cos\theta(4\sin\theta - 1)$$

and thus the critical points are $\theta = \pi/2, -\pi/2$ and $\sin\theta = 1/4$. The graph of $f(\theta)$ is therefore



When $\nu = 0$ and $\theta = 0$ we have $\dot{\theta} = 1$ and thus reflecting and applying stability we have: