**Homework #5**
**MTH 351/651: Fall 2025**
**Instructor: John Gemmer**

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**Directions:** Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a "rough draft"; your submission with well-organized calculations and relevant explanations should be thought of as your "final draft".

## Problems to be completed by all students

**Problem 1.** In this problem we study $2 \times 2$ systems of linear ODEs:

$$\dot{y} = Ay, \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Classify the origin as a a stable/unstable spiral, node, or saddle, and sketch the phase portrait for each of the following cases:

$$A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}.$$

**Problem 2.** Show that

$$x_1(t) = e^{\alpha t} \begin{bmatrix} \cos(\beta t) \\ -\sin(\beta t) \end{bmatrix} \text{ and } x_2(t) = e^{\alpha t} \begin{bmatrix} \sin(\beta t) \\ \cos(\beta t) \end{bmatrix}$$

are two solutions of the linear differential equation:

$$\dot{x} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} x.$$

The principle of linear superposition ensure that all solutions can be written as a linear combination of $x_1$ and $x_2$:

$$x(t) = c_1 x_1(t) + c_2 x_2(t).$$

Write $x(t)$ in the following form:

$$x(t) = ae^{\alpha}(t) \begin{bmatrix} \cos(\beta t + \phi) \\ -\sin(\beta t + \phi) \end{bmatrix},$$

with $a = \sqrt{c_1^2 + c_2^2}$. The parameter $\phi$ is called the phase. Find an expression for the phase depending on $c_1$ and $c_2$.

**Problem 3.** For a $2 \times 2$ matrix $A$ prove that the eigenvalues $\lambda_{1,2}$ of $A$ satisfy:

$$\lambda_{1,2} = \frac{\text{Tr}(A)}{2} \pm \frac{1}{2}\sqrt{(\text{Tr}(A))^2 - 4\det(A)}.$$

**Hint:** You can use the fact that $\text{Tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1 \lambda_2$.

**Problem 4.** Consider the system $\dot{x} = -y$, $\dot{y} = -x$.

(a) Sketch the vector field.

(b) Show that the trajectories of the system are hyperbolas of the form $x^2 - y^2 = C$.

(c) The origin is a saddle; find equations for its stable and unstable manifolds.

**Problem 5.** The motion of a damped harmonic oscillator is described by $m\ddot{x} + b\dot{x} + kx = 0$, where $m, b, k > 0$ are constants.

(a) Rewrite the equation as a two-dimensional linear system.

(b) Classify the fixed point at the origin and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

(c) How do your results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?

## Problems to be completed by graduate students

**Problem 6.** Consider a fixed point $x^*$ of the system $\dot{x} = f(x)$.

- We say $x^*$ is **attracting** if there exists a $\delta > 0$ such that $\lim_{t\to\infty} x(t) = x^*$ whenever $\|x(0) - x^*\| < \delta$.
- We say that $x^*$ is **Liapunov stable** if for each $\varepsilon > 0$, there is a $\delta > 0$ such that $\|x(t) - x^*\| < \varepsilon$ whenever $t \geq 0$ and $\|x(0) - x^*\| < \delta$.
- $x^*$ is **asymptotically stable** if it is both attracting and Liapunov stable.

For each of the following systems, decide whether the origin is attracting, Liapunov stable, asymptotically stable, or none of the above.

(a) $\dot{x} = y$ and $\dot{y} = -4x$.

(b) $\dot{x} = 0$ and $\dot{y} = x$.

(c) $\dot{x} = -x$ and $\dot{y} = -5y$.

(d) $\dot{x} = 2y$ and $\dot{y} = x$.

(e) $\dot{x} = 0$ and $\dot{y} = y$.

(f) $\dot{x} = x$ and $\dot{y} = y$.