

Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a “rough draft”; your submission with well-organized calculations and relevant explanations should be thought of as your “final draft”.

Problems to be completed by all students

Problem 1. The motion of a damped harmonic oscillator is described by $m\ddot{x} + b\dot{x} + kx = 0$, where $m, b, k > 0$ are constants.

- (a) Rewrite the equation as a two-dimensional linear system.
- (b) Classify the fixed point at the origin and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.
- (c) How do your results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?

Problem 2. Consider the system $\ddot{x} = x^3 - x$.

- (a) Write this second order differential equation as a system of first order differential equations.
- (b) Find a conserved quantity for this system.
- (c) Find all the equilibrium and classify them.
- (d) Sketch the phase portrait.
- (e) Find an equation for any heteroclinic or homoclinic orbits.

Problem 3. Consider the system $\ddot{x} = x - x^2$.

- (a) Write this second order differential equation as a system of first order differential equations.
- (b) Find a conserved quantity for this system.
- (c) Find all the equilibrium and classify them.
- (d) Sketch the phase portrait.
- (e) Find an equation for any heteroclinic or homoclinic orbits.

Problem 4. Consider the system $\ddot{\theta} = -(1 + a \cos(\theta))\dot{\theta} - \sin(\theta)$ defined on the circle $\theta \in (-\pi, \pi)$, where $a \geq 0$. This equation models the dynamics of a damped pendulum with nonlinear friction.

- (a) The energy for this system is defined by $E = \frac{1}{2}\dot{\theta}^2 - \cos(\theta)$. Determine the values of a for which E is monotone decreasing, i.e., $\frac{dE}{dt} < 0$.
- (b) Write this second order differential equation as a system of first order differential equations.
- (c) Find all the equilibrium, classify them and sketch all qualitatively different phase portraits that occur as a is varied.

Problem 5. Consider the following system expressed in polar coordinates:

$$\begin{aligned}\dot{r} &= r(1 - r^2), \\ \dot{\theta} &= \mu - \sin(\theta),\end{aligned}$$

where μ is a parameter. Sketch all possible qualitatively different phase portraits that can occur as μ is varied.