Homework #7
MTH 351/651: Fall 2025
Instructor: John Gemmer

Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a "rough draft"; your submission with well-organized calculations and relevant explanations should be thought of as your "final draft".

Problems to be completed by all students

Problem 1. The motion of a damped harmonic oscillator is described by $m\ddot{x} + b\dot{x} + kx = 0$, where $m, b, k > 0$ are constants.

(a) Rewrite the equation as a two-dimensional linear system.

(b) Classify the fixed point at the origin and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the relative sizes of the parameters.

(c) How do your results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?

Problem 2. Consider the system $\ddot{x} = x^3 - x$.

(a) Write this second order differential equation as a system of first order differential equations.

(b) Find a conserved quantity for this system.

(c) Find all the equilibrium and classify them.

(d) Sketch the phase portrait.

(e) Find an equation for any heteroclinic or homoclinic orbits.

Problem 3. Consider the system $\ddot{x} = x - x^2$.

(a) Write this second order differential equation as a system of first order differential equations.

(b) Find a conserved quantity for this system.

(c) Find all the equilibrium and classify them.

(d) Sketch the phase portrait.

(e) Find an equation for any heteroclinic or homoclinic orbits.

Problem 4. Consider the system $\ddot{\theta} = -(1 + a\cos(\theta))\dot{\theta} - \sin(\theta)$ defined on the circle $\theta \in (-\pi, \pi)$, where $a \geq 0$. This equation models the dynamics of a damped pendulum with nonlinear friction.

(a) The energy for this system is defined by $E = \frac{1}{2}\dot{\theta}^2 - \cos(\theta)$. Determine the values of a for which E is monotone decreasing, i.e., $\frac{dE}{dt} < 0$.

(b) Write this second order differential equation as a system of first order differential equations.

(c) Find all the equilibrium, classify them and sketch all qualitatively different phase portraits that occur as a is varied.

Problem 5. Consider the following system expressed in polar coordinates:

$$\dot{r} = r(1 - r^2),$$
$$\dot{\theta} = \mu - \sin(\theta),$$

where μ is a parameter. Sketch all possible qualitatively different phase portraits that can occur as μ is varied.

Homework #7

#1

$m\ddot{x}+b\dot{x}+kx=0.$

Solution:

(a) $m\ddot{x}=-b\dot{x}-kx$

$$\ddot{x}=-\frac{b}{m}\dot{x}-\frac{k}{m}x$$

Letting $v=\dot{x}$, we have

$$\dot{x}=v$$
$$\dot{v}=-\frac{b}{m}v-\frac{k}{m}x.$$

Now, $[v]=L/s$ $[x]=L$ and thus $[b/m]=1/s$, $[k/m]=1/s^2$.
Consequently, there is not a nice way to rescale this problem.
Thus we just set

$$\dot{x}=v \quad ,$$
$$\dot{v}=-av-cx,$$

where $a=b/m$, $c=k/m$.

(b). In matrix form we have

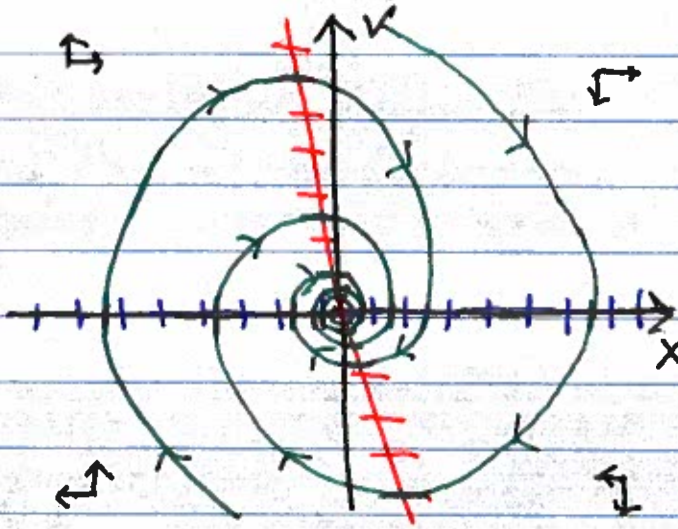
$$\begin{bmatrix}\dot{x}\\ \dot{v}\end{bmatrix}=\begin{bmatrix}0 & 1\\ -a & -c\end{bmatrix}\begin{bmatrix}x\\ v\end{bmatrix}=A\begin{bmatrix}x\\ v\end{bmatrix},$$

where the eigenvalues of A satisfy

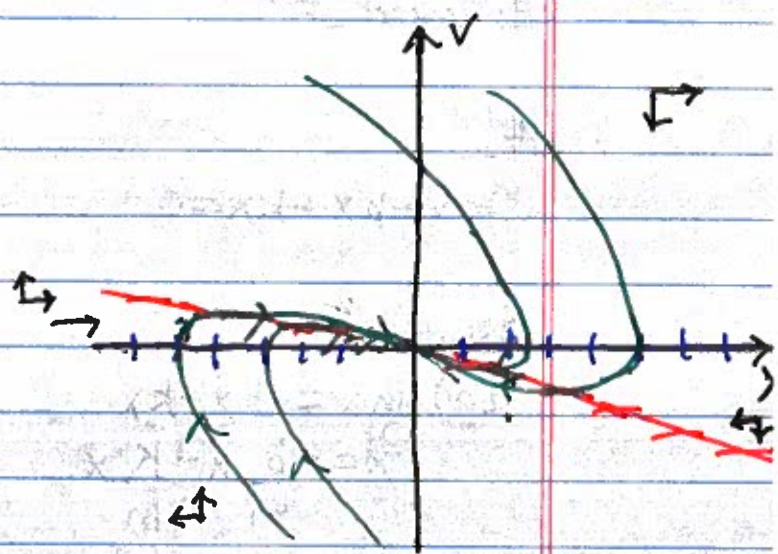
$$\lambda_1+\lambda_2=-c$$
$$\lambda_1\lambda_2=a$$
$$\Rightarrow \lambda_{1,2}=\frac{-c\pm\sqrt{c^2-4a}}{2}.$$

Consequently, $(0,0)$ is a stable spiral if $c^2<4a$ and a stable node if $c^2>4a$.

Case 1: $c^2 < 4a$



(underdamped vibrations)



(overdamped vibrations)

#2.

$$\ddot{x} = x^3 - x$$

Solution:

(a) $\dot{x} = v$

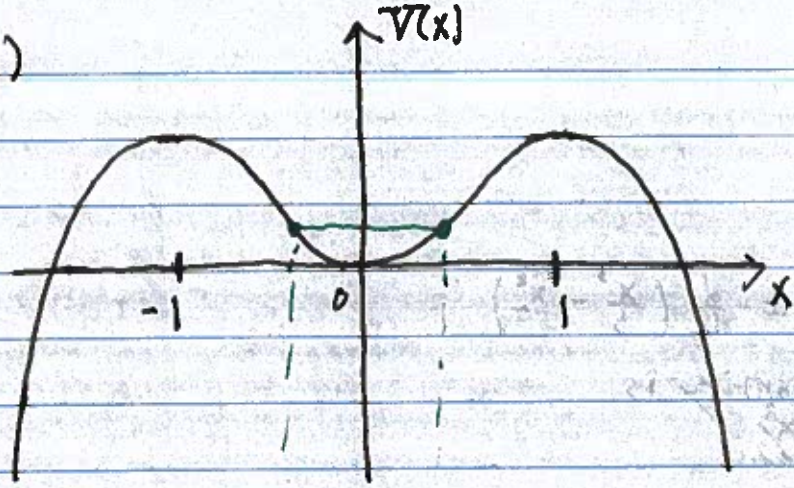
$$\dot{v} = -\frac{d}{dx}\left(\frac{1}{2}x^2 - \frac{1}{4}x^4\right) = -\frac{d}{dx}V(x)$$

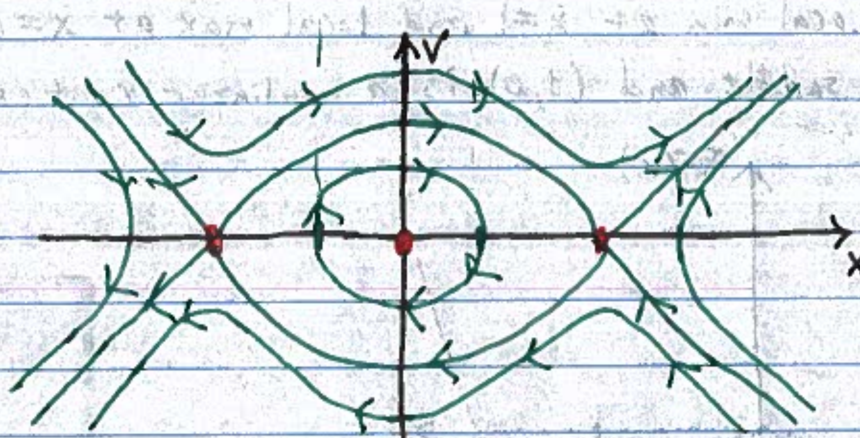
(b) The conserved quantity is

$$E = \frac{1}{2}v^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4.$$

(c) $V(x)$ has a local min at $x=0$ and local maximum at $x=-1,1$ and thus $(0,0)$ is a nonlinear center and $(\pm 1, 0)$ are saddles.

(d)





(e) The heteroclinic orbit connects $(-1,0)$ to $(1,0)$ and due to conservation of energy we have

$$E(1,0) = 1/4 = \frac{1}{2}v^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4$$

$$\Rightarrow v = \pm\sqrt{\frac{1}{2} - x^2 + \frac{1}{2}x^4}.$$

■

#3

$\dot{x} = x - x^2$

Solution:

(a) $\dot{x} = v$

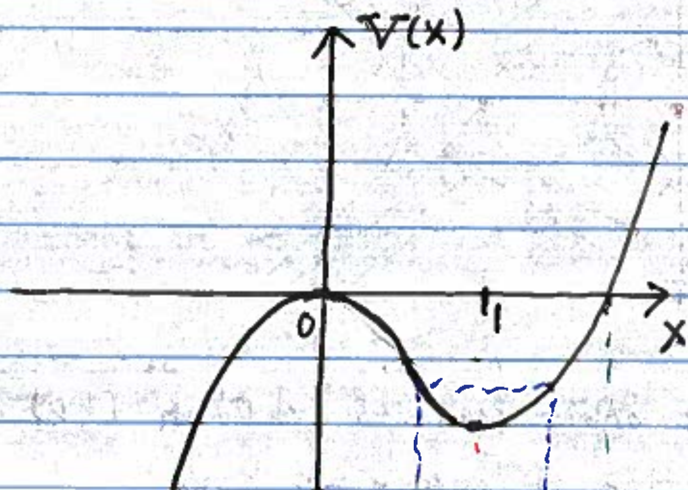
$$\dot{v} = -\frac{d}{dx}\left(\frac{x^3}{3} - \frac{x^2}{2}\right) = -\frac{d}{dx}\left(\frac{x^3}{3} - \frac{x^2}{2}\right)$$

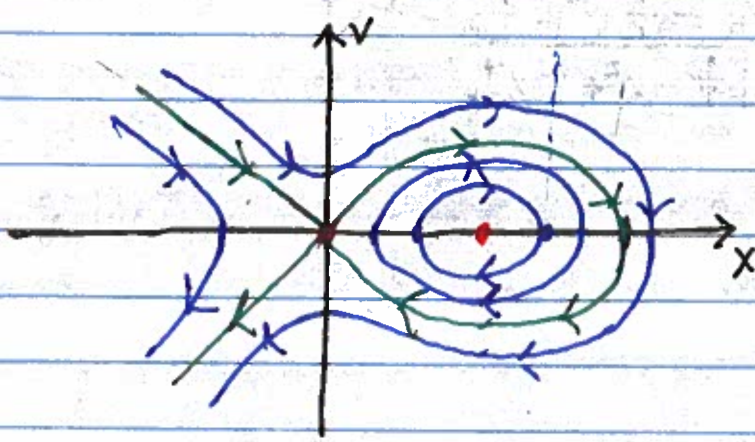
(b) The conserved quantity is

$$E = \frac{1}{2}v^2 + \frac{x^3}{3} - x^2/2.$$

(c) $V(x)$ has a local min at $x=1$ and local max at $x=0$ and thus $(0,0)$ is a saddle and $(1,0)$ is a nonlinear center.

(d)





(e) A homoclinic orbit connects $(0,0)$ with itself and due to conservation of energy we have:

$$0 = \frac{1}{2}v^2 + x^3/3 - x^2/2$$

$$\Rightarrow v = \pm\sqrt{x^2 - \tfrac{2}{3}x^3}$$

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#4

$$\ddot{\theta} = -(1+a\cos(\theta))\dot{\theta} - \sin\theta$$

Solution:

(a) If we let $E = \frac{1}{2}\dot{\theta}^2 - \cos\theta$, we have that

$$\dot{E} = \dot{\theta}\ddot{\theta} + \sin\theta\,\dot{\theta}$$
$$= -(1+a\cos(\theta))\dot{\theta}^2 - \dot{\theta}\sin\theta + \dot{\theta}\sin\theta$$
$$= -(1+a\cos(\theta))\dot{\theta}^2$$

Therefore, if $a<1$ we have that $\dot{E}<0$.

(b) If we let $v = \dot{\theta}$ we have

$$\dot{\theta} = v$$
$$\dot{v} = -(1+a\cos(\theta))v - \sin\theta$$

(c) The fixed points are given by $(0,0)$, $(-\pi,0)$, $(\pi,0)$.

The Jacobian is:

$$J = \begin{bmatrix} 0 & 1 \\ a\sin\theta v - \cos\theta & -(1+a\cos\theta) \end{bmatrix}$$

$J(0,0)$

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & -(1+a) \end{bmatrix}$$

$$\Rightarrow \lambda_{1,2} = \frac{-(1+a) \pm \sqrt{(1+a)^2 - 4}}{2}$$

Therefore, $(0,0)$ is a stable spiral if $a<1$ and a stable node if $a>1$.

$J(0,\pm\pi)$:

$$\begin{bmatrix} 0 & 1 \\ 1 & -(1-a) \end{bmatrix}$$

$$\Rightarrow \lambda_{1,2} = \frac{-(1-a) \pm \sqrt{(1-a)^2 + 4}}{2}.$$

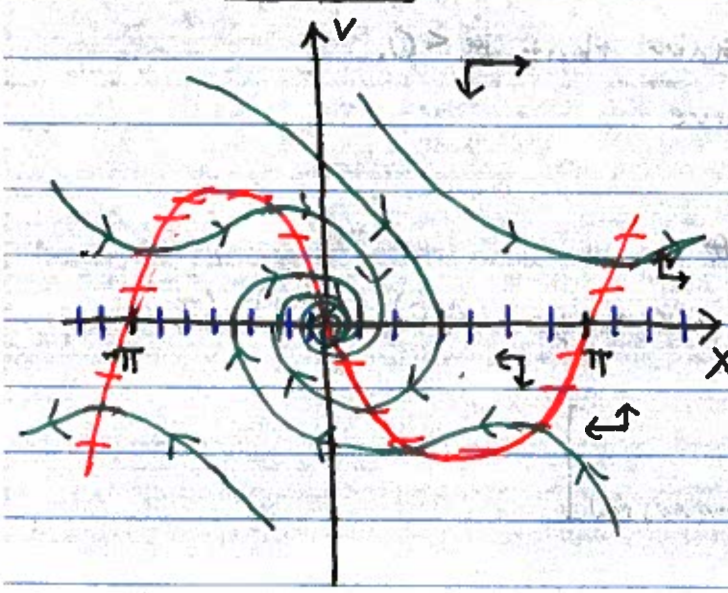
Therefore, $(0, \pm\pi)$ is a saddle for all values of a.

The nullclines for this system are given by

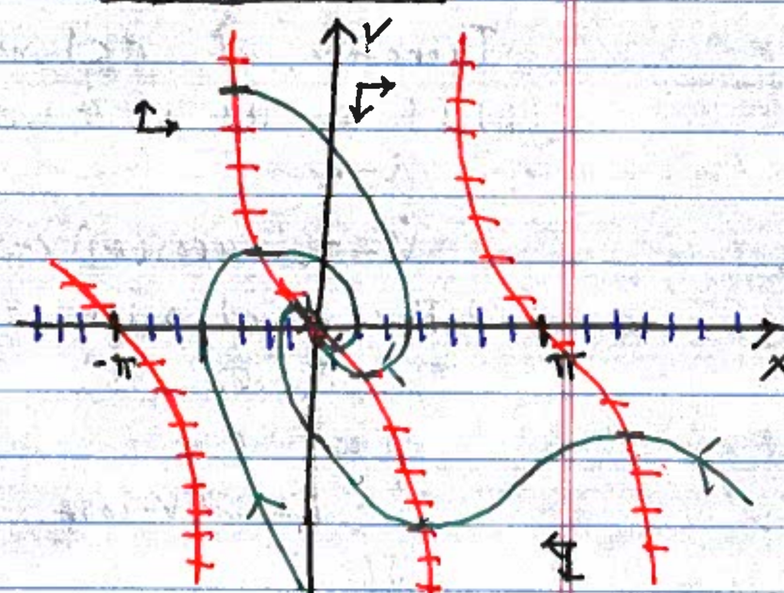
$$\dot{x} = 0 \Rightarrow v = 0$$

$$\dot{v} = 0 \Rightarrow v = -\frac{\sin\theta}{1 + a\cos\theta}$$

Case 1 $(a<1)$:



Case 2 $(a>1)$:



#5

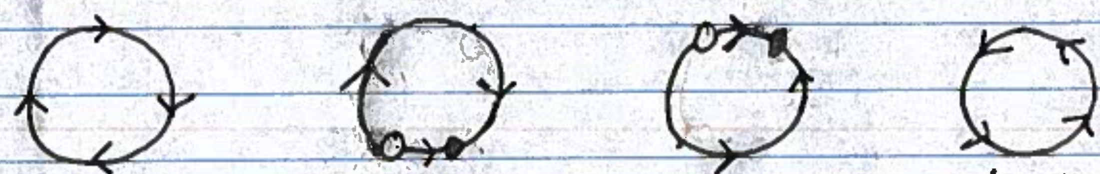
$\dot{r} = r(1-r^2)$

$\dot{\theta} = \mu - \sin\theta$

Solution:

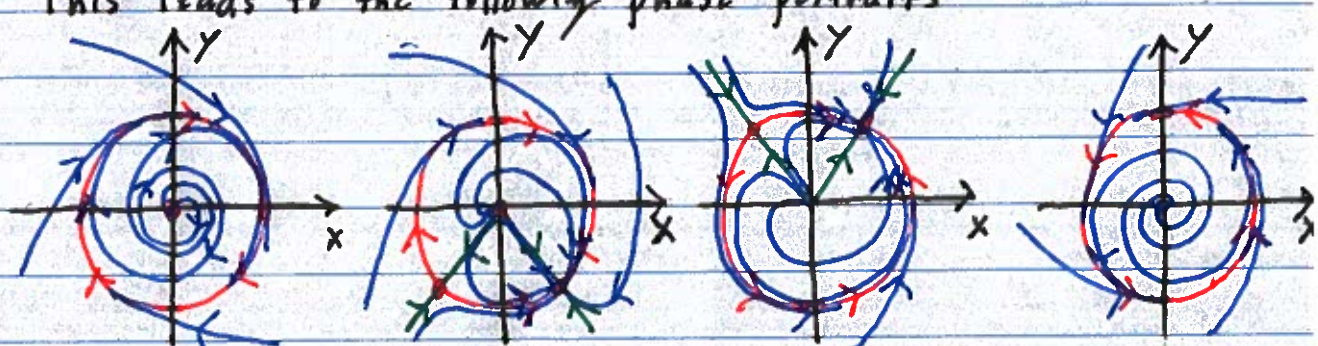
The system is decoupled and the dynamics of the radial coordinate r are complety captured by the following one-dimensional phase portrat

The dynamics in θ can be captured by the following phase portraits on the unit circle



$\mu < -1$ $\qquad -1 < \mu < 0$ $\qquad 0 < \mu < 1$ $\qquad 1 < \mu$

This leads to the following phase portraits



$\mu < -1$ $\qquad -1 < \mu < 0$ $\qquad 0 < \mu < 1$ $\qquad 1 < \mu$

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