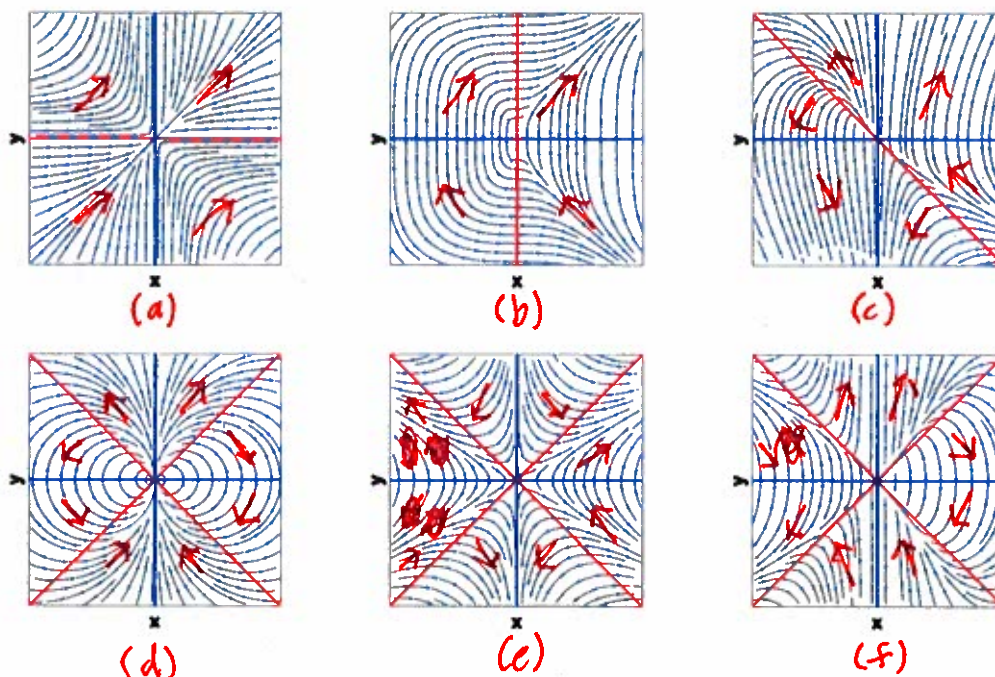


Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a “rough draft”; your submission with well-organized calculations and relevant explanations should be thought of as your “final draft”.

Problems to be completed by all students

Problem 1. The figures below illustrate phase portraits for highly unusual systems. In each figure the blue and red curves correspond to the $\dot{x} = 0$ and $\dot{y} = 0$ nullclines respectively. For each figure, determine the index of the fixed point at the origin.



Problem 2. A closed orbit in phase space encloses S saddles, N nodes, and F spirals. Show that $N + F = 1 + S$.

Problem 3. Use index theory to show that the following system cannot have any closed orbits:

$$\begin{aligned}\dot{x} &= x(4 - y - x^2), \\ \dot{y} &= y(x - 1).\end{aligned}$$

Problem 4. A smooth vector field on the phase plane is known to have exactly three closed orbits. Two of the closed orbits, say C_1 and C_2 , lie inside the third closed orbit C_3 . However, C_1 does not lie inside C_2 , nor vice versa.

- (a) Sketch the arrangement of the three closed orbits.
- (b) Show that there must be at least one fixed point in the region bounded by C_1 , C_2 , and C_3 .

Problem 5. For each of the following problems, decide whether it is a gradient system. If so, find V , sketch the equipotential curves $V = \text{constant}$ and on the same graph sketch the phase portrait.

- (a) $\dot{x} = y + x^2y$ and $\dot{y} = -x + 2xy$
- (b) $\dot{x} = 2x$ and $\dot{y} = 8y$
- (c) $\dot{x} = -2x \exp(x^2 + y^2)$ and $\dot{y} = -2y \exp(x^2 + y^2)$

Homework #8

#1.

(a) $\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \Rightarrow I[\delta] = 0$

(b) $\rightarrow \rightarrow \nearrow \nearrow \rightarrow \Rightarrow I[\delta] = 0$

(c) $\rightarrow \nwarrow \searrow \searrow \nwarrow \rightarrow \Rightarrow I[\delta] = 0$

(d) $\rightarrow \nearrow \searrow \searrow \nearrow \searrow \searrow \nearrow \Rightarrow I[\delta] = 2$

(e) $\rightarrow \searrow \searrow \nearrow \nearrow \searrow \searrow \nearrow \Rightarrow I[\delta] = -2$

(f) $\rightarrow \rightarrow \searrow \nwarrow \nearrow \nearrow \searrow \searrow \nearrow \Rightarrow I[\delta] = 0$

#2.

From index theory we know that

$$-S + N + F = 1$$

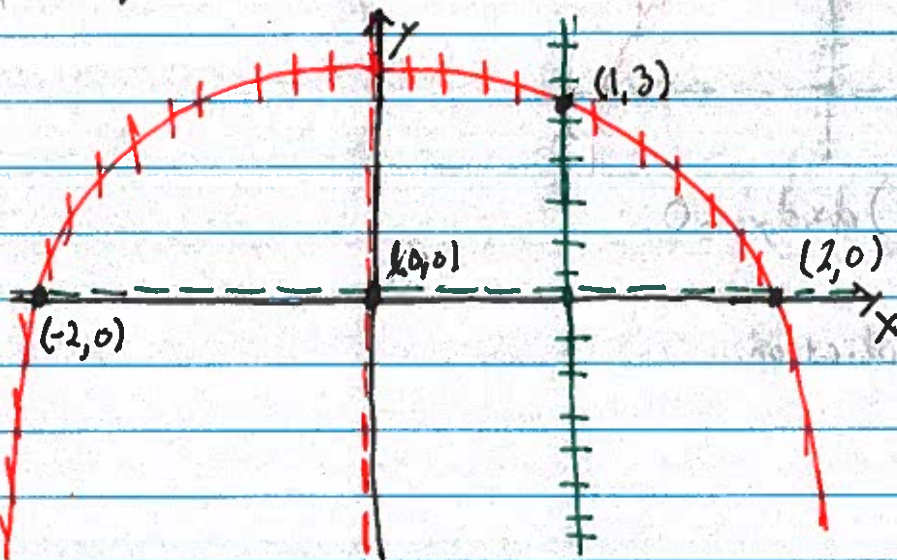
$$\Rightarrow N + F = 1 + S$$

#4.

$$\dot{x} = x(4 - y - x^2)$$

$$\dot{y} = y(x - 1)$$

Sketching the nullclines we have:



No closed orbit can enclose $(-2,0)$, $(0,0)$, $(2,0)$ since the line $y=0$ is invariant. That is, a closed orbit would have to cross the line $y=0$ which would violate uniqueness of solutions. Now,

$$J = \begin{bmatrix} 4-y-3x^2 & -x \\ y & x-1 \end{bmatrix}$$

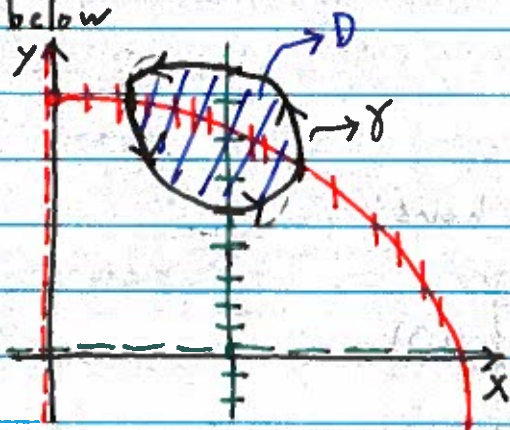
$$\Rightarrow J[(1,3)] = \begin{bmatrix} -2 & -1 \\ 3 & 0 \end{bmatrix} \Rightarrow (1,3) \text{ is a stable spiral and thus has index } 1.$$

I screwed up on this problem, to really eliminate the existence of a limit cycle, you need to apply Dulac's criteria which I outline below.

Let $f(x,y) = \frac{1}{xy}$, and suppose γ is a closed orbit in the first quadrant with outward normal N . Therefore,

$$0 = \oint_{\gamma} f(x,y) \vec{F}(x,y) \cdot N ds = \iint_D \nabla \cdot (f(x,y) \vec{F}(x,y)) dA,$$

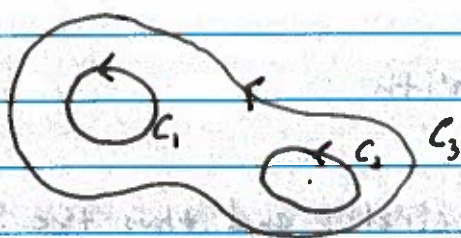
where $\vec{F}$ is the vector field and D is the interior of γ ; see the illustration below



$$\Rightarrow 0 = \iint_D \left(-\frac{2x}{y}\right) dx dy < 0$$

which is a contradiction.

#4



Since $I[C_1]=1$, $I[C_2]=1$ and $I[C_3]=1$, there must exist a fixed point in $C_3 \cap \bar{C}_1 \cap \bar{C}_2$.

#5

(a) $\dot{x} = y + x^2y$

$\dot{y} = -x + 2xy$

$\Rightarrow \frac{\partial \dot{x}}{\partial y} = 1 + x^2, \frac{\partial \dot{y}}{\partial x} = -1 + 2y$

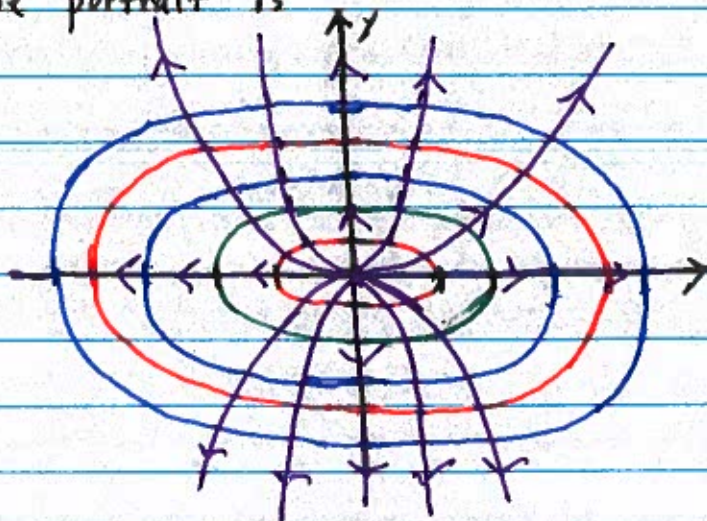
Therefore, this is not a gradient system.

(b) $\dot{x} = 2x$

$\dot{y} = 8y$

This is a gradient system with $V = -x^2 - 4y^2$

The equipotential curves are ellipses and thus the phase portrait is



$$(c) \begin{aligned} \dot{x} &= -2x \exp(x^2 + y^2) \\ \dot{y} &= -2y \exp(x^2 + y^2) \end{aligned}$$

This is a gradient system with
 $V = \exp(x^2 + y^2)$.

The equipotential lines are circles and thus the phase portrait is given by

