Homework #9
MTH 351/651: Fall 2025
Instructor: John Gemmer

Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a "rough draft"; your submission with well-organized calculations and relevant explanations should be thought of as your "final draft".

Problems to be completed by all students

Problem 1. Consider the following system:

$$\dot{x} = -x + 2y^3 - 2y^4,$$
$$\dot{y} = -x - y + xy.$$

(a) Find values of p, q such that the function $L(x, y) = x^p + y^q$ satisfies

$$\frac{d}{dt} L(x(t), y(t)) \leq 0$$

if $(x(t), y(t))$ is a solution to this differential equation.

(b) Prove that this system cannot have a limit cycle.

Problem 2. For the following system, sketch the nullclines, construct a trapping region, and prove that the system has a limit cycle:

$$\dot{x} = x - y - x^3,$$
$$\dot{y} = x + y - y^3.$$

Problem 3. Show that the following system has at least one limit cycle:

$$\dot{x} = x + y - x(x^2 + 2y^2),$$
$$\dot{y} = -x + y - y(x^2 + 2y^2).$$

Problems to be completed by graduate students

Problem 4. Consider the following system

$$\dot{\mathbf{x}} = F(\mathbf{x}),$$

where $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a smooth function.

(a) Suppose $\gamma(t)$ is a limit cycle for this system. If $\mathbf{N}$ denotes the outward normal to γ, show that

$$\int_\gamma F(\gamma(s)) \cdot \mathbf{N} ds = 0.$$

(b) Now, suppose $\nabla \cdot F$ does not change sign in a simply connected region D in $\mathbb{R}^2$. Prove that this system cannot contain any limit cycles in D.

Problem 5. Consider the following system

$$\dot{\mathbf{x}} = F(\mathbf{x}),$$

where $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a smooth function. Prove that if there is a smooth function $h : \mathbb{R}^2 \mapsto \mathbb{R}$ for which $\nabla \cdot (h(\mathbf{x})F(\mathbf{x}))$ is of one sign in a simply connected region D in $\mathbb{R}^2$ then this system cannot contain any limit cycles in D.