Homework #9
MTH 351/651: Fall 2025
Instructor: John Gemmer

Directions: Solve the problems below. Then write or type your solutions clearly. Do not submit disorganized scratch work, only your well-written complete solutions. You should think of any scratch work as you would think of a "rough draft"; your submission with well-organized calculations and relevant explanations should be thought of as your "final draft".

Problems to be completed by all students

Problem 1. Consider the following system:

$$\begin{aligned} \dot{x} &= -x + 2y^3 - 2y^4, \\ \dot{y} &= -x - y + xy. \end{aligned}$$

(a) Find values of p, q such that the function $L(x, y) = x^p + y^q$ satisfies

$$\frac{d}{dt} L(x(t), y(t)) \leq 0$$

if $(x(t), y(t))$ is a solution to this differential equation.

(b) Prove that this system cannot have a limit cycle.

Problem 2. For the following system, sketch the nullclines, construct a trapping region, and prove that the system does not have a limit cycle:

$$\begin{aligned} \dot{x} &= x - y - x^3, \\ \dot{y} &= x + y - y^3. \end{aligned}$$

Problem 3. Show that the following system has at least one limit cycle:

$$\begin{aligned} \dot{x} &= x + y - x(x^2 + 2y^2), \\ \dot{y} &= -x + y - y(x^2 + 2y^2). \end{aligned}$$

Problems to be completed by graduate students

Problem 4. Consider the following system

$$\dot{\mathbf{x}} = F(\mathbf{x}),$$

where $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a smooth function.

(a) Suppose $\gamma(t)$ is a limit cycle for this system. If $\mathbf{N}$ denotes the outward normal to γ, show that

$$\int_\gamma F(\gamma(s)) \cdot \mathbf{N} ds = 0.$$

(b) Now, suppose $\nabla \cdot F$ does not change sign in a simply connected region D in $\mathbb{R}^2$. Prove that this system cannot contain any limit cycles in D.

Problem 5. Consider the following system

$$\dot{\mathbf{x}} = F(\mathbf{x}),$$

where $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a smooth function. Prove that if there is a smooth function $h : \mathbb{R}^2 \mapsto \mathbb{R}$ for which $\nabla \cdot (h(\mathbf{x})F(\mathbf{x}))$ is of one sign in a simply connected region D in $\mathbb{R}^2$ then this system cannot contain any limit cycles in D.

Homework #9

#1.

$$\dot{x} = -x + 2y^3 - 2y^4$$
$$\dot{y} = -x - y + xy$$

Solution:

(a) If $L = x^p + y^q$ then

$$\dot{L} = px^{p-1}\dot{x} + qy^{q-1}\dot{y}$$
$$= px^{p-1}(-x+2y^3-2y^4) + qy^{q-1}(-x-y+xy)$$
$$= -px^p + 2px^{p-1}y^3 - 2px^{p-1}y^4 - qxy^{q-1} - qy^q + qxy^q$$
$$= -px^p - qy^q + 2px^{p-1}y^3 + qxy^q - 2px^{p-1}y^4 - qxy^{q-1}$$

One way to ensure $\dot{L} < 0$ is to require that p, q are even and

$$2px^{p-1}y^3 = qxy^{q-1}$$
$$qxy^q = 2px^{p-1}y^4$$
$$\Rightarrow q = 4, \; p = 2.$$

Therefore, $L = x^2 + y^4$.

(b) Since $\dot{L} \leq 0$ along solution curves and $\dot{L} = 0$ only when $x = y = 0$, it follows that this system cannot have any limit cycles. ∎

#2

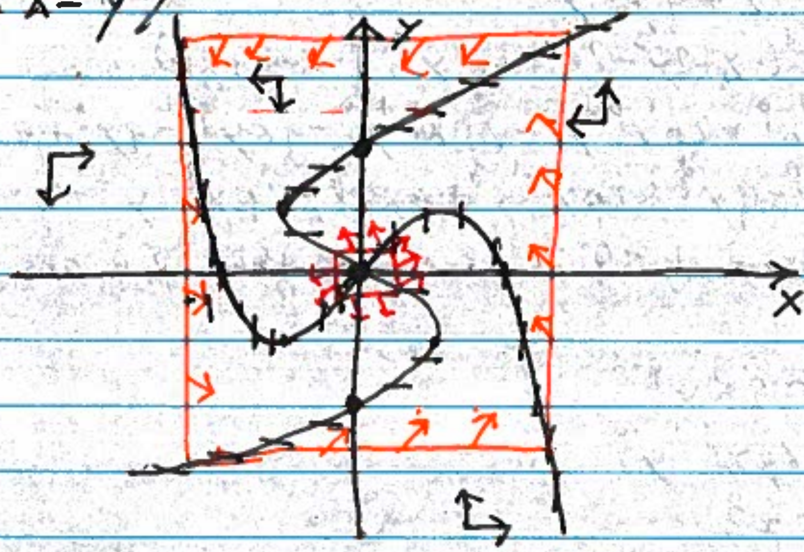
$$\dot{x} = x - y - x^3$$
$$\dot{y} = x + y - y^3$$

Solution:

The nullclines are

$\dot{x}=0$: $y = x - x^3$

$\dot{y}=0$: $x = y^3 - y$



The orange box forms the boundary of a trapping region. Now,

$$J(0,0) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Since $I(0,0)=1$ and $Tr(0,0)=2$ it follows that $(0,0)$ is either an unstable node or an unstable spiral. Therefore, there exists a neighborhood around $(0,0)$ in which the flow pushes away from $(0,0)$; see red curve in above figure. By the Poincare-Bendixon theorem this system has a limit cycle.

$\blacksquare$

#3.

$$\dot{x} = x + y - x(x^2 + 2y^2)$$
$$\dot{y} = -x + y - y(x^2 + 2y^2)$$

Solution:

$$r^2 = x^2 + y^2$$
$$\Rightarrow 2r\dot{r} = 2x\dot{x} + 2y\dot{y}$$
$$\Rightarrow \dot{r} = \frac{x\dot{x} + y\dot{y}}{r}$$
$$= \frac{x^2 + xy - x^2(x^2 + 2y^2) - xy + y^2 - y^2(x^2 + 2y^2)}{r}$$
$$= \frac{x^2 + y^2 - x^4 - 2x^2y^2 - x^2y^2 - 2y^4}{r}$$
$$= \frac{r^2 - r^4\cos^4\theta - 3r^4\cos^2\theta\sin^2\theta - 2r^4\sin^4\theta}{r}$$
$$= r(1 - r^2\cos^4\theta - 3r^4\cos^2\theta\sin^2\theta - 2r^4\sin^4\theta)$$
$$= r(1 - r^2\cos^4\theta - r^4\sin^2\theta(3\cos^2\theta + 2\sin^2\theta))$$
$$= r(1 - r^2\cos^4\theta - r^4\sin^2\theta(\cos^2\theta + 2)$$

If $r > 1$ we have $r^4 > r^2 \Rightarrow -r^2 < -r^4$. Consequently,

$$\dot{r} \leq r(1 - r^4\cos^4\theta - r^4\sin^2\theta\cos^2\theta - 2r^4\sin^2\theta)$$
$$= r(1 - r^4\cos^2\theta(\cos^2\theta + \sin^2\theta) - 2r^4\sin^2\theta)$$
$$= r(1 - r^4\cos^2\theta - 2r^4\sin^2\theta)$$
$$= r(1 - r^4 - r^4\sin^2\theta)$$
$$\leq 0.$$

Consequently, if $r > 1$ it follows that $\dot{r} \leq 0$. Furthermore,

$$J(0,0) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

which implies $(0,0)$ is an unstable spiral. Therefore, by Poincare-Bendixon theorem a limit cycle exists.

#4

(a) Since γ is a limit cycle we know $F(\gamma(s))$ is tangent to γ.
Therefore, $F(\gamma(s)) \cdot \vec{N} = 0$ and thus

$$\int_\gamma F(\gamma(s)) \cdot \vec{N}\, ds = 0.$$

(b) By the divergence theorem:

$$\int_\gamma F(\gamma(s)) \cdot \vec{N}\, ds = \iint_D \nabla \cdot F\, dA = 0.$$

However, if $\nabla \cdot F$ does not change sign then $\iint_D \nabla \cdot F\, dA \neq 0$ unless $\nabla \cdot F$ is identically zero.

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#5

~~proof~~:

Suppose γ is a limit cycle in D with interior D_γ. By the divergence theorem

$$0 = \int_\gamma h(\vec{x}) F(\vec{x}) \cdot N\, ds = \iint_{D_\gamma} \nabla \cdot (h(\vec{x}) F(\vec{x}))\, dA.$$

Consequently, if $\nabla \cdot (h(\vec{x}) F(\vec{x}))$ is of one sign, we obtain a contradiction.

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