Lecture 2: Calculus of Parametric Curves

Motion Along a Line:

An object moves with constant speed along a line through the point (x_0, y_0). Both the x,y-coordinates have constant rate of change. Find the parametric equations for $(x(t), y(t))$.

We know

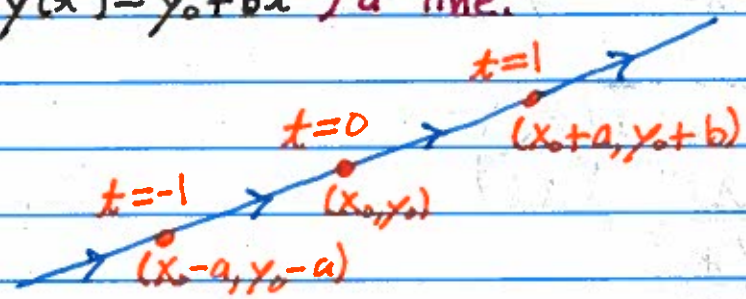
$$a = \frac{dx}{dt}, \quad b = \frac{dy}{dt}$$

The fundamental theorem of calculus implies

$$x(t) - x(0) = \int_0^t \frac{dx}{ds}\,ds = \int_0^t a\,ds = at$$

$$y(t) - y(0) = \int_0^t \frac{dy}{ds}\,ds = \int_0^t b\,ds = bt$$

$$\Rightarrow \left.\begin{aligned} x(t) &= x_0 + at \\ y(t) &= y_0 + bt \end{aligned}\right\} \text{parametric equations for a line.}$$

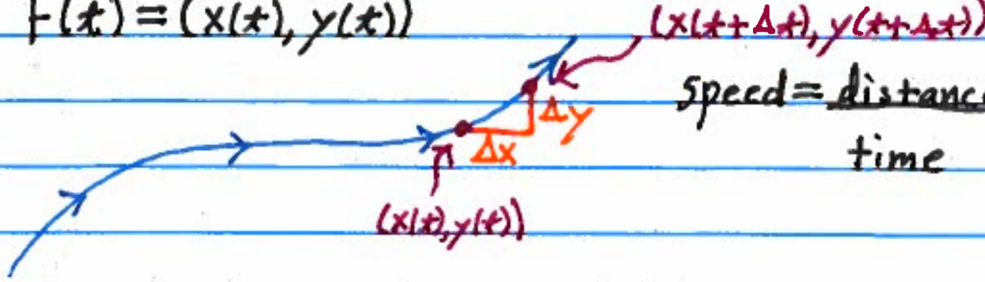


Slope is given by

$$m = \frac{\text{rise}}{\text{run}} = \frac{y(t+\Delta t) - y(t)}{x(t+\Delta t) - x(t)} = \frac{y_0 + b(t+\Delta t) - y_0 - bt}{x_0 + a(t+\Delta t) - x_0 - at} = \frac{b\Delta t}{a\Delta t} = \frac{b}{a}$$

Speed and Velocity

$F(t) = (x(t), y(t))$



$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

$$\Delta x = x(t+\Delta t) - x(t)$$
$$\Delta y = y(t+\Delta t) - y(t)$$

$$\Rightarrow \text{speed} = \frac{\text{distance}}{\text{time}} \approx \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta t}$$

$$\Rightarrow \text{speed} \approx \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

$$\Rightarrow \text{speed} \approx \sqrt{\left(\frac{x(t+\Delta t) - x(t)}{\Delta t}\right)^2 + \left(\frac{y(t+\Delta t) - y(t)}{\Delta t}\right)^2}$$

$$\Rightarrow v = \lim_{\Delta t \to 0} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

$$\Rightarrow v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Example

A particle moves in the xy-plane with

$$x = 2t^3 - 9t^2 + 12t, \quad t \geq 0$$
$$y = 3t^4 - 16t^3 + 18t^2$$

Where t is time.

a) At what times is the particle stopped?

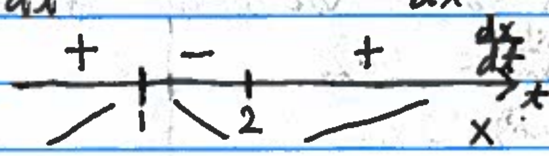
b) At what times is the particle moving parallel to the x-axis?

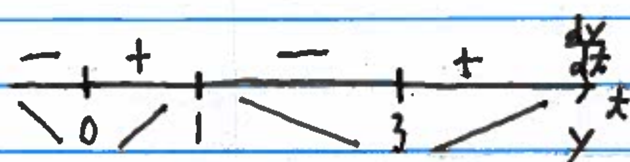
$$\frac{dx}{dt} = 6t^2 - 18t + 12, \quad \frac{dy}{dt} = 12t^3 - 48t^2 + 36t$$

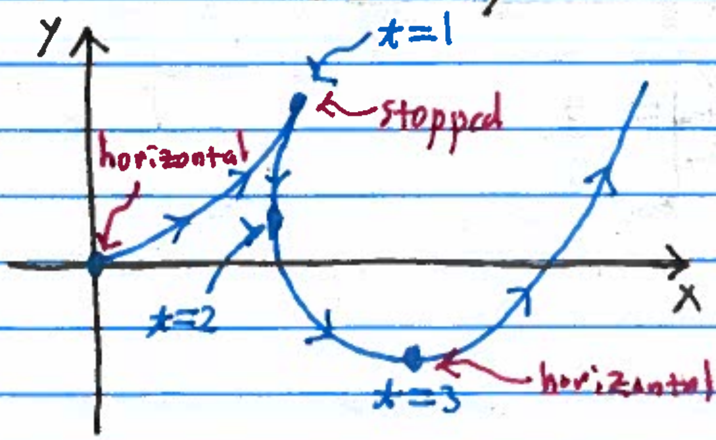
$$\Rightarrow \frac{dx}{dt} = 6(t^2 - 3t + 2), \quad \frac{dy}{dt} = 12t(t^2 - 4t + 3)$$

$$\Rightarrow \frac{dx}{dt} = 6(t-1)(t-2), \quad \frac{dy}{dt} = 12t(t-1)(t-3)$$

$$\frac{dx}{dt} = 0 \Rightarrow t = 1, 2, \quad \frac{dy}{dt} = 0 \Rightarrow t = 1, 3$$



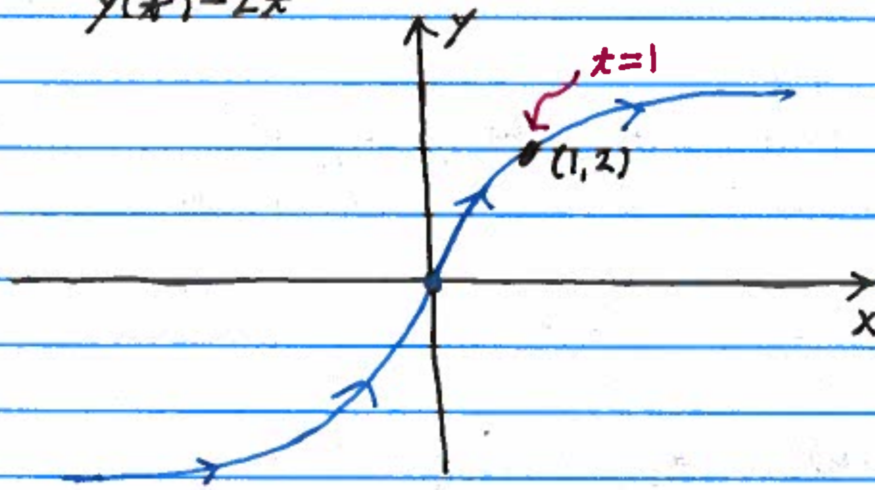




Tangent Lines

Find the tangent line at $(1,2)$ to the curve defined by

$$x(t) = t^3$$
$$y(t) = 2t$$



Two methods:

1. $\left.\frac{dx}{dt}\right|_{t=1} = 3t^2\big|_{t=1} = 3, \quad \left.\frac{dy}{dt}\right|_{t=1} = 2$

The tangent line is given by, if we center at $t=1$:

$$x(t) - 1 = 3(t-1) \Rightarrow x(t) = 3t - 2$$
$$y(t) - 2 = 2(t-1) \Rightarrow y(t) = 2t$$

If we center at $t=0$:

$$x(t) = 3t + 1$$
$$y(t) = 2t + 2$$

2. $\left.\frac{dy}{dx}\right|_{t=1} = \left.\frac{dy}{dt}\frac{dt}{dx}\right|_{t=1} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=1} = \frac{3}{2}$

$$\Rightarrow y - 2 = \tfrac{3}{2}(x-1)$$
$$\Rightarrow y = \tfrac{3}{2}(x-1) + 2$$

What is the concavity at $(1,2)$?

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d}{dx}\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{d}{dx}\frac{2}{3t^2} = \frac{dt}{dx}\frac{d}{dt}\left(\frac{2}{3t^2}\right) = \frac{-4/3\, t^{-3}}{\frac{dx}{dt}} = -\frac{4}{3t^5}.$$

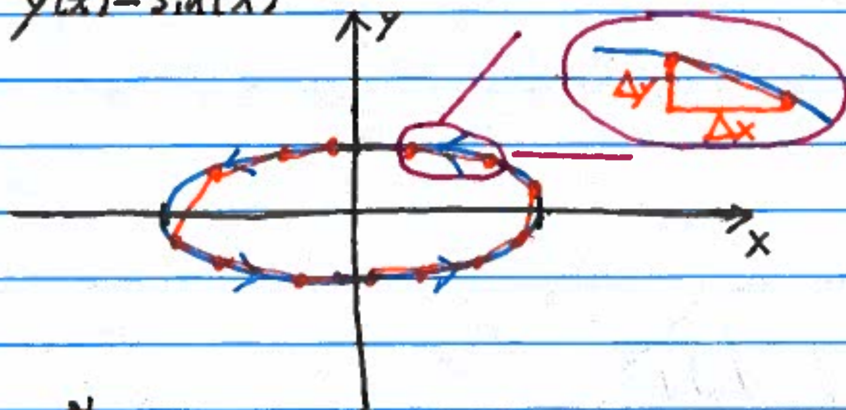
$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=1} = -\frac{4}{3}.$$

Arc Length

Find the circumference of the ellipse:

$x(t) = 2\cos(t)$

$y(t) = \sin(t)$



$$L \approx \sum_{i=1}^{N} \sqrt{\Delta x^2 + \Delta y^2}$$

$$= \sum_{i=1}^{N} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t$$

$$\Rightarrow L = \lim_{N\to\infty} \sum_{i=1}^{N} \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\,\Delta t \quad (\Delta t \to 0)$$

$$= \int_0^{\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{4\sin^2(t) + \cos^2(t)}\,dt$$

$$\approx 9.69$$

The quantity

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

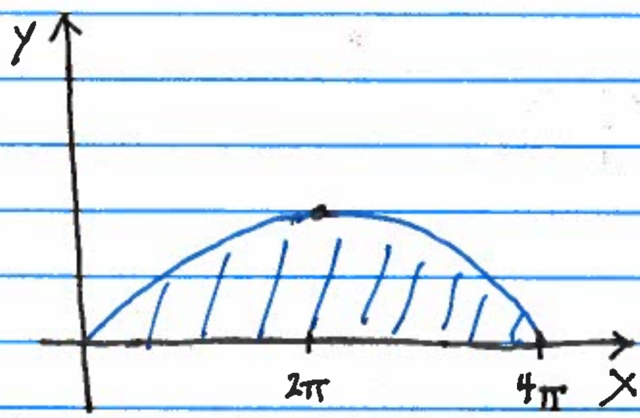
is called the arc-length element.

Example:

Find the area under the curve

$x = 2(\theta - \sin\theta), \quad 0 \le \theta \le 2\pi$

$y = 1 - \cos\theta$



$A = \int_0^{4\pi} y\,dx$

$dx = \frac{dx}{d\theta}\,d\theta = 2(1-\cos\theta)\,d\theta$

$\Rightarrow A = \int_0^{2\pi} 2(1-\cos\theta)(1-\cos\theta)\,d\theta$

$= 2\int_0^{2\pi} 2(1 - 2\cos\theta + \cos^2\theta)\,d\theta$

$= 2\int_0^{2\pi} 2\left(1 - 2\cos\theta + \frac{1+\cos(2\theta)}{2}\right)d\theta$

$= 2\int_0^{2\pi} 3/2\,d\theta$

$= 2 \cdot \frac{3}{2}\theta \Big|_0^{2\pi}$

$= 6\pi$