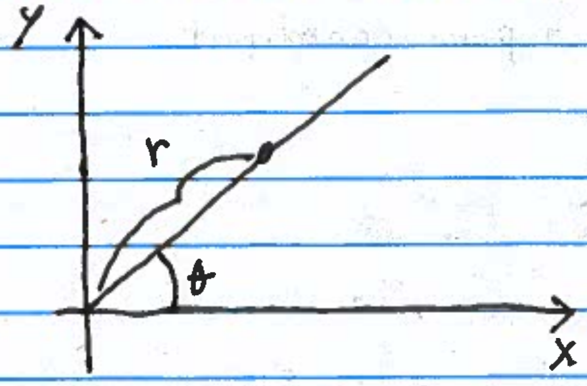
Lecture #3: Polar Coordinates



$$x = r\cos\theta \quad \Rightarrow \quad r^2 = x^2 + y^2$$
$$y = r\sin\theta \qquad\quad \tan\theta = \frac{y}{x}$$

Example:

1. Give Cartesian coordinates for the point with polar coordinates $r=7$, $\theta=\pi/3$.

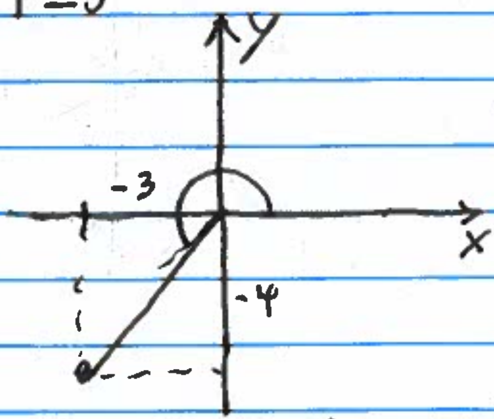
   $x = 7\cos(\pi/3) = 7/2$

   $y = 7\sin(\pi/3) = 7\cdot\sqrt{3}/2$

2. Convert to polar coordinates $x=-3$, $y=-4$.

   $x^2+y^2=25$

   $\Rightarrow r=5$



$\Rightarrow \theta = \pi + \tan^{-1}(4/3)$

Example:

Write the equation $y=1$ in polar coordinates.

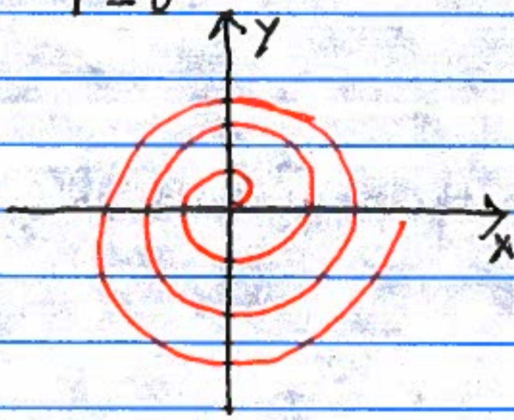
$$\Rightarrow r\sin\theta = 1$$

$$\Rightarrow r = \frac{1}{\sin\theta} = \csc\theta.$$

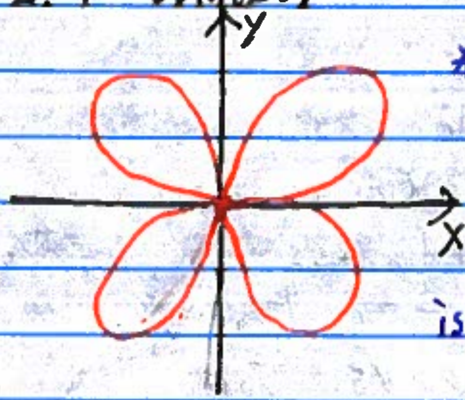
Example:

Sketch the following curves defined in polar coordinates.

1. $r=\theta$

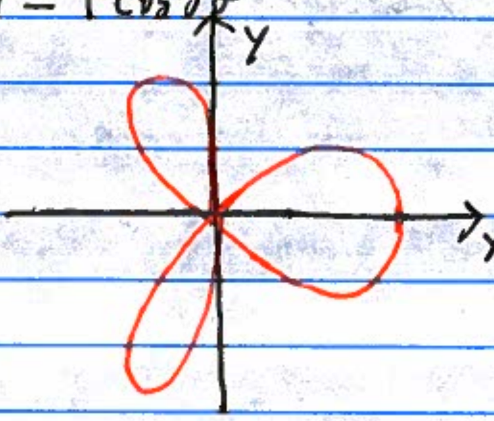


2. $r=3\sin(2\theta)$

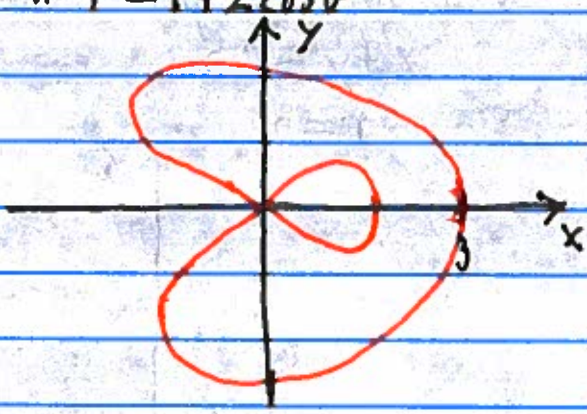


* If $r<0$, the relationship
$x = r\cos\theta$
$y = r\sin\theta$
is still true.

3. $r=4\cos 3\theta$



4. $r=1+2\cos\theta$



Slope and Concavity

$$r = f(\theta)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

However,

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

$$\Rightarrow \frac{dx}{d\theta} = \frac{df}{d\theta}\cos\theta - f(\theta)\sin(\theta)$$

$$\frac{dy}{d\theta} = \frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta$$

Therefore,

$$\frac{dy}{dx} = \frac{\frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta}$$

Now,

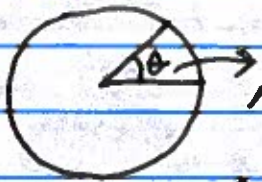
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{d\theta}{dx}\frac{d}{d\theta}\frac{dy}{dx}$$

$$= \frac{d}{d\theta}\left(\frac{\frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta}\right) \Big/ \left(\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta\right.$$

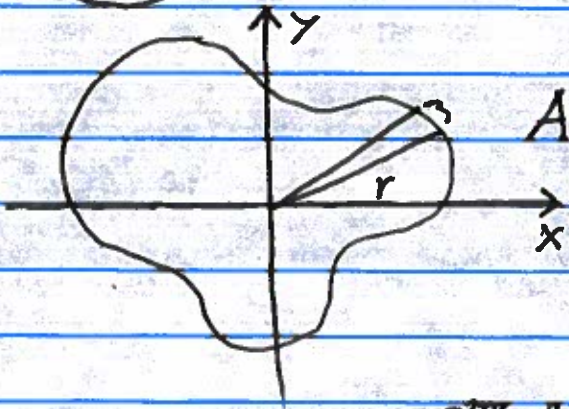
$$= \frac{\left(\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta\right)\left(\frac{d^2f}{d\theta^2}\sin\theta + \frac{df}{d\theta}\cos\theta + \frac{df}{d\theta}\cos\theta - f(\theta)\sin\right. - \left(\frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta\right)\left(\frac{d^2f}{d\theta^2}\cos\theta - 2\frac{df}{d\theta}\sin\theta - f(\theta)\cos\theta\right.}{\left(\frac{df}{d\theta}\cos\theta - f(\theta)\sin(\theta)\right)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2\left(\frac{df}{d\theta}\right)^2 - f(\theta)\frac{d^2f}{d\theta^2} + f(\theta)^2}{\left(\frac{df}{d\theta}\cos\theta - f(\theta)\sin(\theta)\right)^3} = \frac{f(\theta)\frac{d^2f}{d\theta^2} - 2\left(\frac{df}{d\theta}\right)^2 - f(\theta)^2}{\left(f(\theta)\sin(\theta) - \frac{df}{d\theta}\cos\theta\right)^3}$$

Area:



$$\text{Area} = \frac{\theta}{2\pi}\pi r^2 = \frac{\theta r^2}{2}$$



$$\text{Area of slice} \approx \frac{\Delta\theta\, r(\theta)^2}{2}$$

$$\Rightarrow \text{Total Area} = \int_0^{2\pi} \frac{r^2}{2}\, d\theta$$

Example:

Find the area enclosed inside the curve

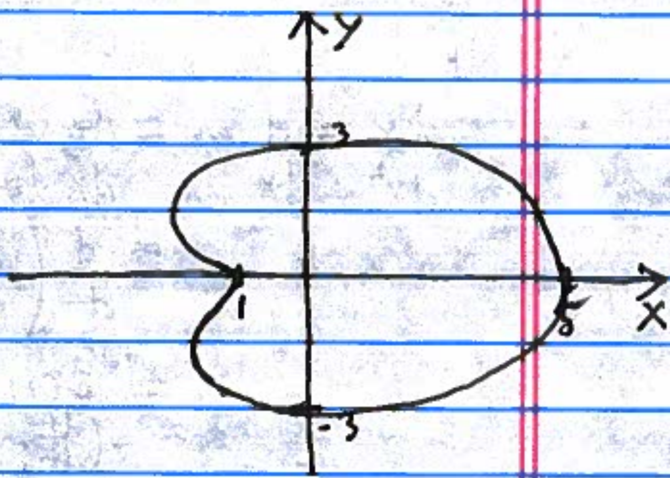
$$r(\theta) = 3 + 2\cos\theta$$

$$\text{Area} = \int_0^{2\pi} \frac{(3+2\cos\theta)^2}{2}\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{2} + \frac{6\cos\theta}{4} + 2\cos^2\theta\right) d\theta$$

$$= 2\pi\left(\frac{9}{2}\right) + 2\pi$$

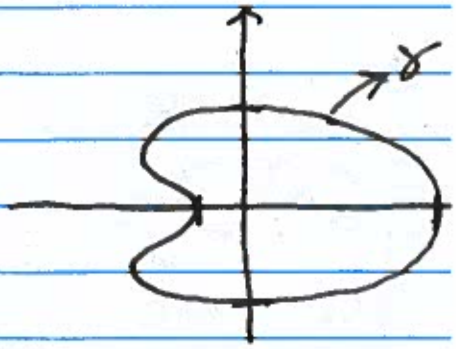
$$= 11\pi$$



Example:

Find the length of the curve

$r(\theta) = 3 + 2\cos\theta$



$$L = \int_\gamma ds = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dx}{d\theta}\right)^2}\, d\theta$$

$x(\theta) = (3 + 2\cos\theta)\cos\theta = 3\cos\theta + 2\cos^2\theta$

$y(\theta) = (3 + 2\cos\theta)\sin\theta = 3\sin\theta + 2\cos\theta\sin\theta$

$\Rightarrow \frac{dx}{d\theta} = -3\sin\theta + 4\cos\theta\sin\theta$

$\frac{dy}{d\theta} = 3\cos\theta - 2\sin^2\theta + 2\cos^2\theta$

$$\Rightarrow L = \int_0^{2\pi} \sqrt{\begin{matrix} 9\sin^2\theta - 12\cos\theta\sin^2\theta + 16\cos^2\theta\sin^2\theta + 9\cos^2\theta \\ -12\cos\theta\sin^2\theta + 12\cos^3\theta + 4\sin^4\theta - 8\sin^2\theta\cos^2\theta + 4\cos^4\theta \end{matrix}}\, d\theta$$