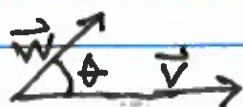


Lecture #6: Vector Products

The Dot Product

$$\vec{v} \cdot \vec{w}$$

1. Geometric: $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos \theta$



2. Algebra: $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

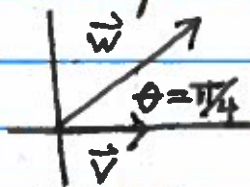
Note: $\vec{v} \cdot \vec{w} = \text{number}$

Example:

Suppose $\vec{v} = \vec{i}$ and $\vec{w} = 2\vec{i} + 2\vec{j}$. Compute $\vec{v} \cdot \vec{w}$.

(i) $\|\vec{v}\| = 1$

$$\|\vec{w}\| = 2\sqrt{2}$$



$$\Rightarrow \vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos \theta = 1 \cdot 2\sqrt{2} \cos(\pi/4) = 2$$

(ii) $\vec{v} \cdot \vec{w} = \langle 1, 0, 0 \rangle \cdot \langle 2, 2, 0 \rangle = 1 \cdot 2 + 0 \cdot 2 + 0 \cdot 0 = 2.$

Consequence:

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

2. $\vec{v} \cdot (\lambda \vec{w}) = \lambda \vec{v} \cdot \vec{w}$

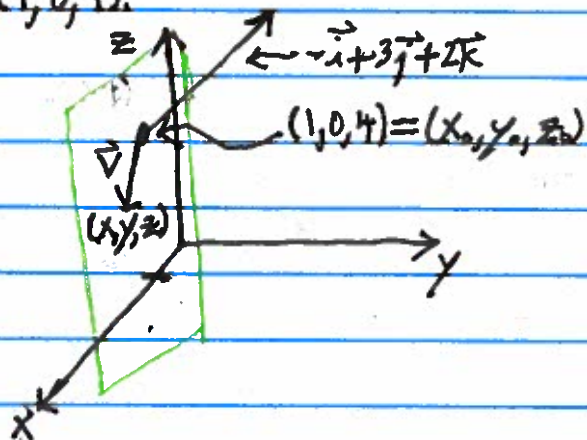
3. $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$

4. $\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}$

5. $\vec{v} \perp \vec{w}$ if and only if $\vec{v} \cdot \vec{w} = 0$

Example:

Find the equation of the plane orthogonal to $-\vec{i} + 3\vec{j} + 2\vec{k}$ and passing through $(1, 0, 4)$.



$$\vec{V} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$$

$$\Rightarrow \vec{V} \cdot \langle -1, 3, 2 \rangle = 0$$

$$\Rightarrow -(x - x_0) + 3(y - y_0) + 2(z - z_0) = 0$$

$$\Rightarrow -(x - 1) + 3y + 2(z - 4) = 0$$

Components and Projections:

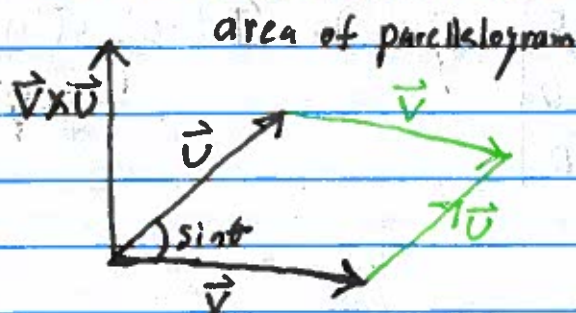


$$\text{proj}_U \vec{V} = \|\vec{V}\| \cos \theta \cdot \frac{\vec{U}}{\|\vec{U}\|} \xrightarrow{\text{direction}} = \frac{\|\vec{V}\| \cdot \|\vec{U}\| \cos \theta}{\|\vec{U}\|^2} \cdot \vec{U} = \frac{\vec{V} \cdot \vec{U}}{\|\vec{U}\|^2} \cdot \vec{U} \quad \text{No trig needed.}$$

$$\text{comp}_U \vec{V} = \frac{\vec{V} \cdot \vec{U}}{\|\vec{U}\|}$$

Cross Products

$$\vec{v} \times \vec{u} = \|\vec{u}\| \|\vec{v}\| \sin \theta \cdot \hat{n} \rightarrow \text{Unit vector orthogonal to both } \vec{u} \text{ and } \vec{v}.$$



Find $\hat{n}$?

$$\langle n_1, n_2, n_3 \rangle$$

$$\hat{n} \cdot \vec{u} = 0$$

$$\hat{n} \cdot \vec{v} = 0$$

$$\|\hat{n}\| = 1$$

$$\begin{aligned} \Rightarrow n_1 u_1 + n_2 u_2 + n_3 u_3 &= 0 \\ n_1 v_1 + n_2 v_2 + n_3 v_3 &= 0 \\ n_1^2 + n_2^2 + n_3^2 &= 1 \end{aligned} \left. \vphantom{\begin{aligned} \Rightarrow n_1 u_1 + n_2 u_2 + n_3 u_3 &= 0 \\ n_1 v_1 + n_2 v_2 + n_3 v_3 &= 0 \\ n_1^2 + n_2^2 + n_3^2 &= 1 \end{aligned}} \right\} \text{Solve for } n!!$$

Determinants

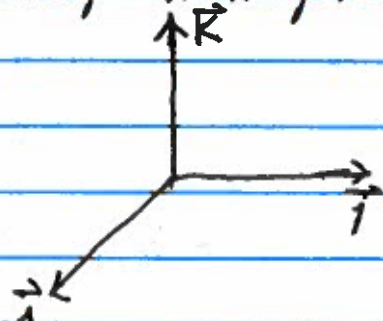
$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}(u_2 v_3 - u_3 v_2) - \hat{j}(u_1 v_3 - u_3 v_1) + \hat{k}(u_1 v_2 - u_2 v_1)$$

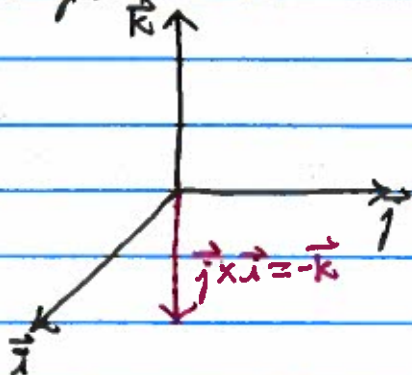
Example:

1. $\vec{i} \times \vec{j} = \|\vec{i}\| \cdot \|\vec{j}\| \sin(\frac{\pi}{2}) \vec{k} = \vec{k}$



$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0-1 \cdot 0)\vec{i} - (1 \cdot 0 - 0)\vec{j} + (1 \cdot 1 - 0)\vec{k} = \vec{k}$$

2. $\vec{j} \times \vec{i} = -\vec{k}$



3. $\vec{v} \times \vec{v} = \vec{0}$

$$\vec{v} \times \vec{v} = \|\vec{v}\| \cdot \|\vec{v}\| \sin(0) \vec{n} = \vec{0}$$

4. $\langle 2, 1, -2 \rangle \times \langle 3, 0, 1 \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \vec{i} \cdot (1-0) - \vec{j} \cdot (2+6) + (2 \cdot 0 - 3)\vec{k} = \vec{i} - 3\vec{k}$$

Properties:

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

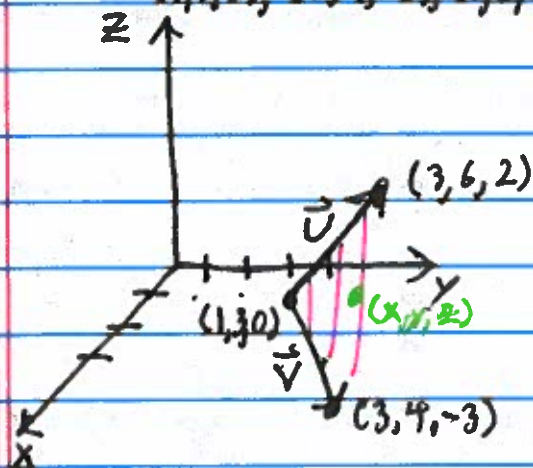
2. $(\lambda \vec{u}) \times \vec{v} = \lambda(\vec{u} \times \vec{v}) = \vec{u} \times (\lambda \vec{v})$

3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

Example:

Find the equation of the plane through

$(1, 3, 0), (3, 4, -3), (3, 6, 2)$



$$\vec{u} = \langle 2, 3, 2 \rangle, \vec{v} = \langle 2, 1, -3 \rangle$$

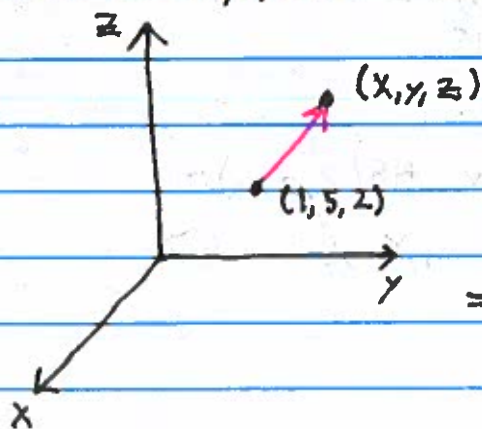
$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 2 \\ 2 & 1 & -3 \end{vmatrix} = -11\vec{i} + 10\vec{j} - 4\vec{k}$$

$$\Rightarrow (\vec{u} \times \vec{v}) \cdot \langle x-1, y-3, z-0 \rangle = 0$$

$$\Rightarrow -11(x-1) + 10(y-3) - 4z = 0$$

Example:

Find the parametric equation of a line through $(1, 5, 2)$ parallel to $\langle 4, 3, 7 \rangle$.



$$\langle x-1, y-5, z-2 \rangle = t \langle 4, 3, 7 \rangle$$

Displacement
vector

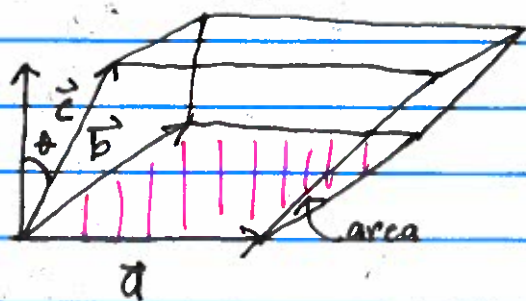
All possible
vectors parallel

$$\Rightarrow x-1=4t \Rightarrow x=1+4t$$

$$y-5=3t \quad y=5+3t$$

$$z-2=7t \quad z=2+7t$$

Volume of Parallelepiped:



$$\text{Area} = \|\vec{a} \times \vec{b}\|$$

$$\text{height} \cdot \text{Area} = \text{Volume}$$

$$h = \|\vec{c}\| \cos \theta$$

$$\Rightarrow V = \|\vec{a} \times \vec{b}\| \cdot \|\vec{c}\| \cos \theta$$

$$\Rightarrow V = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\Rightarrow V = \begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot \langle \vec{c}_1, \vec{c}_2, \vec{c}_3 \rangle = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

* Determinants measure volume!!

Example:

Find a normal vector to the plane

$$3x + 2y + 5z = 15$$



$$\vec{V} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

$$3x_1 + 2y_1 + 5z_1 = 15$$

$$3x_2 + 2y_2 + 5z_2 = 15$$

$$\Rightarrow 3(x_2 - x_1) + 2(y_2 - y_1) + 5(z_2 - z_1) = 0$$

$$\Rightarrow \langle 3, 2, 5 \rangle \cdot \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = 0$$

$$\Rightarrow \vec{n} = \langle 3, 2, 5 \rangle$$