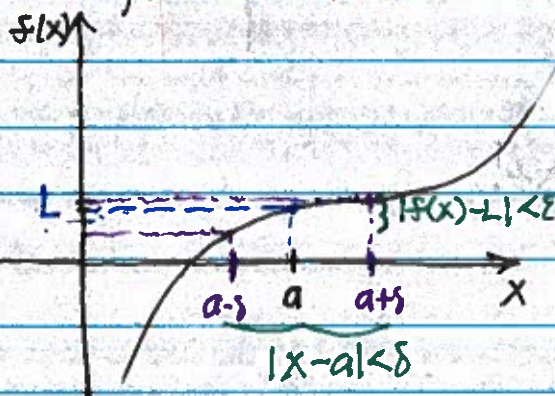
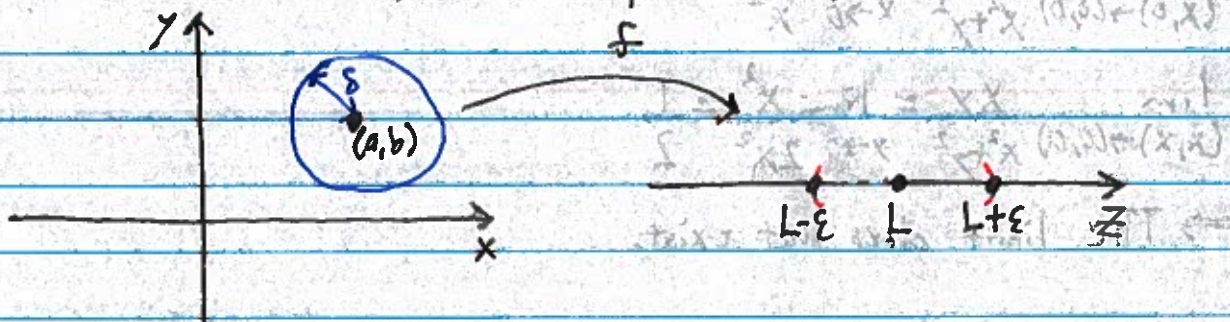


Lecture #8: Limits and Continuity

1. $\lim_{x \rightarrow a} f(x) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - L| < \varepsilon$



2. $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\sqrt{(x-a)^2 + (y-b)^2} < \delta$ implies $|f(x,y) - L| < \varepsilon$.



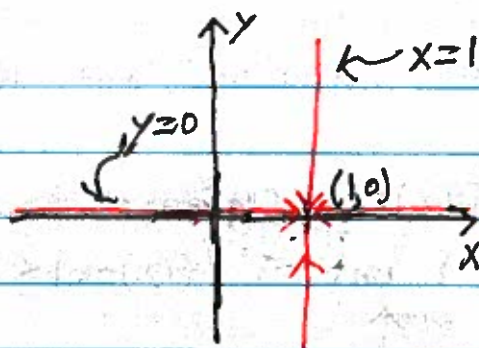
Examples

- $\lim_{(x,y) \rightarrow (0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = -3$ (Continuity)
- $\lim_{(x,y) \rightarrow (3,4)} \sqrt{x^2 + y^2} = 5$ (Continuity)
- $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y) \rightarrow (0,0)} \frac{x(x-y)(\sqrt{x} + \sqrt{y})}{x - y} = \lim_{(x,y) \rightarrow (0,0)} x(\sqrt{x} + \sqrt{y}) = 0$
(Removable Discontinuity)

$$4. \lim_{(x,y) \rightarrow (1,0)} \frac{y}{x+y-1} = ??$$

$$\lim_{(1,y) \rightarrow (1,0)} \frac{y}{x+y-1} = \lim_{y \rightarrow 0} \frac{y}{y} = 1$$

$$\lim_{(x,0) \rightarrow (1,0)} \frac{y}{x+y-1} = \lim_{x \rightarrow 1} \frac{0}{x-1} = 0$$



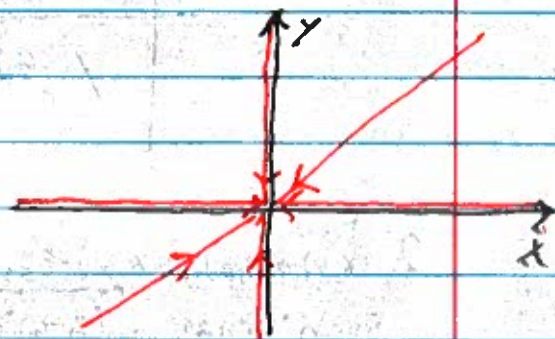
⇒ The limit does not exist!

$$5. \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2} = ?$$

$$\lim_{(0,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2} = \lim_{y \rightarrow 0} \frac{0}{y^2} = 0$$

$$\lim_{(x,0) \rightarrow (0,0)} \frac{xy}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{0}{x^2} = 0$$

$$\lim_{(x,x) \rightarrow (0,0)} \frac{xy}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

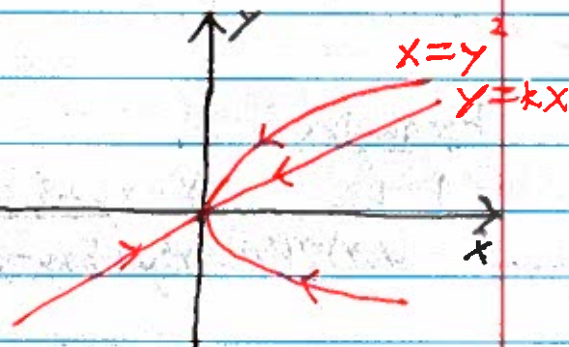


⇒ The limit does not exist.

$$6. \lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2+y^4} = ?$$

$$\lim_{(x,kx) \rightarrow (0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x \rightarrow 0} \frac{k^3 x^3}{x^2 + k^4 x^4} = \lim_{x \rightarrow 0} \frac{x k^3}{1 + k^4 x^2} = 0$$

$$\lim_{(y^2,y) \rightarrow (0,0)} \frac{xy^2}{x^2+y^4} = \lim_{y \rightarrow 0} \frac{y^4}{2y^4} = \frac{1}{2}$$



⇒ The limit does not exist!!

Theorem - If $|f(x,y) - L| \leq g(x)$ and $\lim_{(x,y) \rightarrow (a,b)} g(x,y) = 0$ then $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$.

Example:

1. $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2} = ??$

$$\left| \frac{x^2 y}{x^2 + y^2} \right| \leq \left| \frac{x^2 y}{x^2} \right| = |y|$$

Since $\lim_{(x,y) \rightarrow (0,0)} |y| = 0$ it follows that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2} = 0.$$

2. $\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)^2 \ln(x)}{(x-1)^2 + y^2}$

$$\left| \frac{(x-1)^2 \ln(x)}{(x-1)^2 + y^2} \right| \leq \left| \frac{(x-1)^2 \ln(x)}{(x-1)^2} \right| = |\ln(x)|$$

Since $\lim_{(x,y) \rightarrow (1,0)} |\ln(x)| = 0$ it follows that

$$\lim_{(x,y) \rightarrow (1,0)} \frac{(x-1)^2 \ln(x)}{(x-1)^2 + y^2} = 0.$$