

Bonus Lecture: How to generate random variables

Suppose X has a uniform distribution on $[0, 1]$, i.e.,

$$X \sim \begin{cases} 1, & x \in [0, 1] \\ 0, & \text{o.w.} \end{cases}$$

How can we find f such that $Y = f(X)$ has the density

$$Y \sim \begin{cases} \lambda e^{-\lambda y}, & y > 0 \\ 0, & y \leq 0 \end{cases}$$

For $[a, b] \subseteq [0, 1]$, we want

$$P(X \leq y) = \int_0^y \lambda e^{-\lambda x} dx = 1 - e^{-\lambda y}$$

$$\Rightarrow P(f(X) \leq y) = 1 - e^{-\lambda y}$$

$$\Rightarrow P(X \leq f^{-1}(y)) = 1 - e^{-\lambda y} \text{ or } P(X \geq f^{-1}(y)) = 1 - e^{-\lambda y}$$

$$\Rightarrow \int_0^{f^{-1}(y)} dx = 1 - e^{-\lambda y} \text{ or } \int_{f^{-1}(y)}^1 dx = 1 - e^{-\lambda y}$$

$$\Rightarrow f^{-1}(y) = 1 - e^{-\lambda y} \text{ or } 1 - f^{-1}(y) = 1 - e^{-\lambda y}$$

$$\Rightarrow f^{-1}(y) = 1 - e^{-\lambda y} \text{ or } f^{-1}(y) = e^{-\lambda y}$$

$$\Rightarrow \ln(1 - f^{-1}(y)) = -\lambda y \text{ or } y = \frac{\ln(f^{-1}(y))}{-\lambda}$$

$$\Rightarrow f(x) = \frac{\ln(1-x)}{-\lambda} \text{ or } f(x) = \frac{-\ln(x)}{\lambda}$$