

MTH 383/683

Fall 2026

Exam 1U

09/25/26

Name (Print):

key

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **If you use a “fundamental theorem” you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as “Short Answer” can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	15	
2	15	
3	20	
4	15	
5	15	
6	20	
Total:	100	

Do not write in the table to the right.

1. (15 points) **Short Answer:** Determine if the following statement is correct (C) or incorrect (I). Just circle C or I. No need to show any work. In order for a statement to be correct it must be true in all cases.

(C) I If $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ are mutually disjoint, then

$$P(A \cup B) = P(A) + P(B)$$

C (I) If $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ are independent, then

$$P(A \cap B) = 0.$$

(C) I If $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ are mutually disjoint events with $P(A) > 0$ and $P(B) > 0$, then A and B cannot be independent.

$$0 = P(A \cap B) \neq P(A)P(B) \neq 0.$$

(C) I If $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ are independent, and $P(B) \neq 0$ then

$$P(A|B) = P(A).$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

C (I) If $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ satisfy $A \subset B$ and $P(B) > 0$ then

$$P(A|B) = P(A).$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} \neq P(A)$$

2. (15 points) Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ satisfy

$$P(A) = \frac{1}{3}, \quad P(B) = \frac{1}{2}, \quad \text{and } P(A \cap B) = \frac{1}{4}.$$

(a) (5 points) Compute $P(A \cup B)$.

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{1}{3} + \frac{1}{2} - \frac{1}{4} \\ &= \frac{7}{12} \end{aligned}$$

(b) (5 points) Compute $P(A|B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

(c) (5 points) Compute $P(A^c \cap B)$.

$$\begin{aligned} P(A^c \cap B) &= P(B) - P(A \cap B) \\ &= \frac{1}{2} - \frac{1}{4} \\ &= \frac{1}{4} \end{aligned}$$

3. (20 points) Suppose X is a random variable that is uniformly distributed on the interval $[0, 2]$ and let $Y = \sqrt{X}$.

- (a) (5 points) **Short Answer:** Write down the probability density $f(x)$ that corresponds to the random variable X .

$$f(x) = \begin{cases} \frac{1}{2}, & 0 \leq x \leq 2 \\ 0, & \text{o.w.} \end{cases}$$

- (b) (10 points) Find the cumulative distribution function (CDF) of Y .

$$\begin{aligned} F(y) = P(Y \leq y) &= \begin{cases} 0, & y < 0 \\ P(\sqrt{X} \leq y), & 0 \leq y \leq \sqrt{2} \\ 1, & y > \sqrt{2} \end{cases} \Rightarrow F(y) = \begin{cases} 0, & y < 0 \\ y^2/2, & 0 \leq y \leq \sqrt{2} \\ 1, & y > \sqrt{2} \end{cases} \\ &= \begin{cases} 0, & y < 0 \\ P(X \leq y^2), & 0 \leq y \leq \sqrt{2} \\ 1, & y > \sqrt{2} \end{cases} \\ &= \begin{cases} 0, & y < 0 \\ \int_0^{y^2} \frac{1}{2} dy, & 0 \leq y \leq \sqrt{2} \\ 1, & y > \sqrt{2} \end{cases} \end{aligned}$$

- (c) (5 points) Find the probability density function (PDF) of Y .

$$f(y) = \frac{dF}{dy} = \begin{cases} 0, & y < 0 \text{ or } y > \sqrt{2} \\ y, & 0 \leq y \leq \sqrt{2} \end{cases}$$

4. (15 points) Let X be an exponential random variable with the following density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases},$$

where $\lambda > 0$.

- (a) (5 points) Compute $P(1 < X < 3)$.

$$P(1 < X < 3) = \int_1^3 \lambda e^{-\lambda x} dx = e^{-\lambda} - e^{-3\lambda}$$

- (b) (5 points) Compute $P(2 < X < 4 \text{ and } 1 < X < 3)$.

$$P(2 < X < 4, 1 < X < 3) = \int_2^3 \lambda e^{-\lambda x} dx = e^{-2\lambda} - e^{-3\lambda}$$

- (c) (5 points) Compute $P(2 < X < 4 \mid 1 < X < 3)$.

$$P(2 < X < 4 \mid 1 < X < 3) = \frac{e^{-2\lambda} - e^{-3\lambda}}{e^{-\lambda} - e^{-3\lambda}}$$

5. (15 points) Let $\vec{X} = (X, Y)$ have the following joint probability density

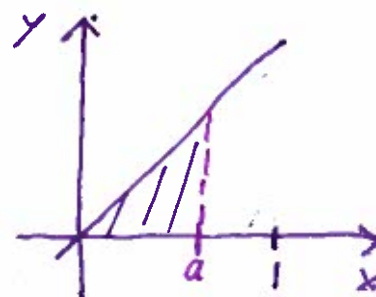
$$f(x, y) = \begin{cases} 2(x+y) & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- (a) (5 points) **Short Answer:** Are X and Y independent random variables?

No

- (b) (5 points) Find the CDF of the random variable X .

$$\begin{aligned} F_X(a) = P(X \leq a) &= \begin{cases} 0, & a < 0 \\ \int_0^a \int_0^x 2(x+y) dy dx, & 0 < a < 1 \\ 1, & a > 1 \end{cases} \\ &= \begin{cases} 0, & a < 0 \\ \int_0^a (2x^2 + x^2) dx, & 0 < a < 1 \\ 1, & a > 1 \end{cases} \\ &= \begin{cases} 0, & a < 0 \\ a^3, & 0 < a < 1 \\ 1, & a > 1 \end{cases} \end{aligned}$$



- (c) (5 points) Find the marginal density $f_X(x)$.

$$f_X(x) = \frac{dF_X}{dx} = \begin{cases} 0, & x < 0 \text{ or } x > 1 \\ 3x^2, & 0 < x < 1 \end{cases}$$

6. (20 points) Let X and Y be independent random variables with moment generating functions

$$\phi_X(t) = e^{t^2/2} \text{ and } \phi_Y(t) = \frac{1}{1-t},$$

respectively, and let $Z = X + Y$.

- (a) (5 points) **Short Answer:** Write down the definition of the moment generating function $\phi_Z(t)$ of the random variable Z .

$$\phi_Z(t) = \mathbb{E}[e^{tZ}]$$

- (b) (5 points) Find the moment generating function of Z .

$$\begin{aligned}\phi_Z(t) &= \mathbb{E}[e^{tZ}] = \mathbb{E}[e^{tX} e^{tY}] = \mathbb{E}[e^{tX}] \mathbb{E}[e^{tY}] \\ \Rightarrow \phi_Z(t) &= \frac{e^{t^2/2}}{1-t}\end{aligned}$$

- (c) (5 points) Compute $\mathbb{E}[Z]$.

$$\begin{aligned}\phi'_X(t) &= t e^{t^2/2}, \quad \phi'_Y(t) = \frac{1}{(1-t)^2} \\ \Rightarrow \phi'_X(0) &= 0, \quad \phi'_Y(0) = 1 \\ \Rightarrow \mathbb{E}[X] &= 0, \quad \mathbb{E}[Y] = 1 \\ \Rightarrow \mathbb{E}[Z] &= \mathbb{E}[X] + \mathbb{E}[Y] = 1.\end{aligned}$$

- (d) (5 points) Compute $\mathbb{E}[XY]$.

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \mathbb{E}[Y] = 0.$$

MTH 383/683
Fall 2026
Exam 1G
09/25/26

Name (Print):

Key

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- If you use a “fundamental theorem” you must indicate this and explain why the theorem may be applied.
- Organize your work, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- Short answer questions: Questions labeled as “Short Answer” can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit. An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Problem	Points	Score
1	15	
2	20	
3	15	
4	15	
5	15	
6	20	
Total:	100	

Do not write in the table to the right.

1. (15 points) **Short Answer:** Determine if the following statement is correct (C) or incorrect (I). Just circle C or I. No need to show any work. In order for a statement to be correct it must be true in all cases.

(C) I If $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ are mutually disjoint, then

$$P(A \cup B) = P(A) + P(B)$$

C (I) If $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ are independent, then

$$P(A \cap B) = 0.$$

(C) I If $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ are mutually disjoint events with $P(A) > 0$ and $P(B) > 0$, then A and B cannot be independent.

(C) I If $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ are independent, and $P(B) \neq 0$ then

$$P(A|B) = P(A).$$

C (I) If $(\Omega, \mathcal{F}, P)$ is a probability space and $A, B \in \mathcal{F}$ satisfy $A \subset B$ and $P(B) > 0$ then

$$P(A|B) = P(A).$$

2. (20 points) Suppose X is a random variable that is uniformly distributed on the interval $[0, 2]$ and let $Y = \sqrt{X}$.

- (a) (5 points) **Short Answer:** Write down the probability density $f(x)$ that corresponds to the random variable X .

$$f \sim \begin{cases} 1/2, & 0 < x < 2 \\ 0, & \text{o.w.} \end{cases}$$

- (b) (10 points) Find the cumulative distribution function (CDF) of Y .

$$\begin{aligned} F(y) = P(Y \leq y) &= \begin{cases} 0, & y < 0 \\ P(\sqrt{X} \leq y), & 0 \leq y \leq \sqrt{2} \\ 1, & y > \sqrt{2} \end{cases} \Rightarrow F(y) = \begin{cases} 0, & y < 0 \\ y^2/2, & 0 \leq y \leq \sqrt{2} \\ 1, & y > \sqrt{2} \end{cases} \\ &= \begin{cases} 0, & y < 0 \\ P(X \leq y^2), & 0 \leq y \leq \sqrt{2} \\ 1, & y > \sqrt{2} \end{cases} \\ &= \begin{cases} 0, & y < 0 \\ \int_0^{y^2} \frac{1}{2} dy, & 0 \leq y \leq \sqrt{2} \\ 1, & y > \sqrt{2} \end{cases} \end{aligned}$$

- (c) (5 points) Find the probability density function (PDF) of Y .

$$f(y) = \frac{dF}{dy} = \begin{cases} 0, & y < 0 \\ y, & 0 \leq y \leq \sqrt{2} \\ 0, & y > \sqrt{2} \end{cases}$$

3. (15 points) Let X be an exponential random variable with the following density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases},$$

where $\lambda > 0$.

(a) (5 points) Compute $P(1 < X < 3)$.

$$P(1 < X < 3) = \int_1^3 \lambda e^{-\lambda x} dx = e^{-\lambda} - e^{-3\lambda}$$

(b) (5 points) Compute $P(2 < X < 4 \text{ and } 1 < X < 3)$.

$$P(2 < X < 4, 1 < X < 3) = \int_2^3 \lambda e^{-\lambda x} dx = e^{-2\lambda} - e^{-3\lambda}$$

(c) (5 points) Compute $P(2 < X < 4 \mid 1 < X < 3)$.

$$P(2 < X < 4 \mid 1 < X < 3) = \frac{e^{-2\lambda} - e^{-3\lambda}}{e^{-\lambda} - e^{-3\lambda}}$$

4. (15 points) Let $\vec{X} = (X, Y)$ have the following joint probability density

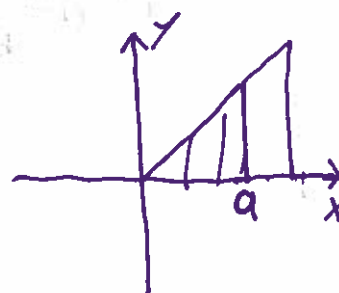
$$f(x, y) = \begin{cases} 2(x+y) & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}.$$

- (a) (5 points) **Short Answer:** Are X and Y independent random variables?

No.

- (b) (5 points) Find the CDF of the random variable X .

$$\begin{aligned} F_X(a) &= P(X < a) = \begin{cases} 0, & a < 0 \\ \int_0^a \int_0^y 2(x+y) dy dx, & 0 < a < 1 \\ 1, & a > 1 \end{cases} \\ &= \begin{cases} 0, & a < 0 \\ a^3, & 0 < a < 1 \\ 1, & a > 1 \end{cases} \end{aligned}$$



- (c) (5 points) Find the marginal density $f_X(x)$.

$$f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 0, & a < 0 \text{ or } a > 1 \\ 3a^2, & 0 < a < 1 \end{cases}$$

5. (15 points) Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $A_1, A_2, \dots \in \mathcal{F}$ be mutually disjoint events satisfying

$$P(A_n) = \frac{1}{2^n}.$$

Define

$$B_n = \bigcup_{k=1}^n A_k \quad \text{and} \quad B = \bigcup_{k=1}^{\infty} A_k.$$

- (a) (5 points) Compute $P(B_n)$ and **briefly** justify your answer. **Hint:** Recall that for $r \neq 1$,

$$\sum_{k=1}^n ar^k = \frac{ar(1-r^n)}{1-r}.$$

$$P(B_n) = \sum_{k=1}^n P(A_k) = \sum_{k=1}^n \frac{1}{2^k} = \frac{1}{2} \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = \frac{1}{2} (1 - (\frac{1}{2})^n) = 1 - (\frac{1}{2})^n$$

- (b) (5 points) Compute $P(B)$ and **briefly** justify your answer.

By construction, $B_n \subseteq B_{n+1}$ and thus by continuity of probability

$$P(B) = P\left(\bigcup_{k=1}^{\infty} A_k\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n \rightarrow \infty} P(B_n) = \lim_{n \rightarrow \infty} 1 - (\frac{1}{2})^n = 1.$$

- (c) (5 points) The "lim-sup" of a sequence of events $A_1, A_2, \dots \in \mathcal{F}$ is the event defined by

$$\limsup_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Compute

$$P\left(\limsup_{n \rightarrow \infty} A_n\right)$$

and **briefly** justify your answer.

If we let $C_n = \bigcup_{k=n}^{\infty} A_k$ it follows that $C_n \subseteq C_{n+1}$. Also,

$$P(C_n) = 1 - P(B_n). \text{ Therefore, by continuity of probability}$$

$$P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\right) = P\left(\bigcap_{n=1}^{\infty} C_n\right) = \lim_{n \rightarrow \infty} P(C_n) = \lim_{n \rightarrow \infty} 1 - P(B_n) = 0.$$

6. (20 points) Let X be a Gaussian random variable on a probability space $(\Omega, \mathcal{F}, P)$ with mean $\mu = 0$ and variance σ^2 .

(a) (5 points) **Short Answer:** For a positive random variable Y on $(\Omega, \mathcal{F}, P)$ write down Markov's inequality.

$$P(Y > a) \leq \frac{1}{a} \mathbb{E}[Y], \text{ for } a > 0.$$

(b) (5 points) For $n \in \mathbb{N}$, prove Chebyshev's inequality:

$$P(|X| > n\sigma) \leq \frac{1}{n^2}.$$

$$P(|X| > n\sigma) = P(X^2 > n^2\sigma^2) \leq \frac{1}{n^2\sigma^2} \mathbb{E}[X^2] = \frac{1}{n^2}$$

(c) (5 points) For $n \in \mathbb{N}$, prove that for all $t > 0$

$$P(X > n\sigma) \leq e^{-tn\sigma + \frac{\sigma^2}{2}t^2}.$$

Hint: The moment generating function of X is given by $\phi_X(t) = e^{\sigma^2 t^2/2}$.

$$P(X > n\sigma) = P(e^{tX} > e^{tn\sigma}) \leq e^{-tn\sigma} \mathbb{E}[e^{tX}] = e^{-tn\sigma + \frac{\sigma^2}{2}t^2}$$

(d) (5 points) Using the result in part (c), prove that

$$P(X > n\sigma) \leq e^{-n^2/2}.$$

Hint: Think about how you would make the upper bound in part (c) as small as possible. **Remark:** The point of this problem is to demonstrate how much the bound on Chebyshev's inequality can be improved given knowledge of the distribution.

Minimizing $-tn\sigma + \frac{\sigma^2}{2}t^2$, the minimum value occurs when

$$\frac{d}{dt}(-tn\sigma + \frac{\sigma^2}{2}t^2) = -n\sigma + \sigma^2 t = 0,$$

Thus the minimum occurs when $t = n/\sigma$ yielding

$$P(X > n\sigma) \leq e^{-n^2 + n^2/2} = e^{-n^2/2}.$$