Homework #1, Due Date: 09/04/26
MTH 383/683: Fall 2026
Instructor: John Gemmer

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1 Problems for all Students

Problem 1. Prove the following defining properties of a probability space $(\Omega, \mathcal{F}, P)$.

(a) If $A, B \in \mathcal{F}$ satisfy $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$,

(b) For all $A \in \mathcal{F}$, $P(A^c) = 1 - P(A)$,

(c) For all $A, B \in \mathcal{F}$, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,

(d) If $A, B \in \mathcal{F}$ and $A \subseteq B$ then $P(A) \leq P(B)$.

Problem 2. Let Ω be a sample space with a finite number of elements and let $\mathcal{F}$ be the set of all subsets of Ω. Prove that if we define $P : \mathcal{F} \mapsto [0,1]$ by $P(A) = \frac{|A|}{|\Omega|}$ then P satisfies the three defining properties of a probability. Note, in this problem $|\cdot|$ denotes the cardinality of a set.

Problem 3. Let ρ_X be the distribution of a random variable X on some probability space $(\Omega, \mathcal{F}, P)$. Show that ρ_X satisfies the three defining properties of a probability on $\mathbb{R}$.

Problem 4. Consider the probability space $(\Omega, \mathcal{F}, P)$. Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of events in $\mathcal{F}$ such that $P(A_n) = 1$ for all $n \geq 1$ and let $B = \bigcap_{n=1}^{\infty} A_n$.

(a) Prove that $B \in \mathcal{F}$,

(b) Prove that $P(B) = 1$.

Problem 5. Let X be a random variable on $(\Omega, \mathcal{F}, P)$ that is uniformly distributed on $[-1, 1]$ and let $Y = X^2$.

(a) Find the CDF of Y and plot its graph.

(b) Find the PDF of Y and plot its graph.

Problem 6. Let $(\Omega, \mathcal{F}, P)$ be some probability space. Recall that two events $A, B \in \mathcal{F}$ are independent if $P(A \cap B) = P(A)P(B)$.

(a) Prove that if A and B are independent, then A^c and B are also independent.

(b) Prove that if A and B are independent, then A^c and B^c are also independent.

(c) Prove that if A and B are independent, then

$$P(A \cup B) = 1 - (1 - P(A))(1 - P(B)).$$

2 Problems for Graduate Students

Problem 7. Hint: You should work through the proof of Lemma 1.4 in the text before doing this problem. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_n \in \mathcal{F}$ be a sequence of events. Prove that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n).$$

Problem 8. Hint: You should work through the proof of Lemma 1.4 in the text before doing this problem. Let $(\Omega, \mathcal{F}, P)$ be a probability space. We say a sequence of events A_n converges to A as $n \to \infty$, written as $A_n \to A$, if for all $\omega \in A$ there exists $N(\omega) \in \mathbb{N}$ such that for all $n \geq N(\omega)$, $\omega \in A_n$ if and only if $\omega \in A$.

(a) Prove that $A_n \to A$ if and only if for all $\omega \in \Omega$, $\lim_{n\to\infty} \mathbb{1}_{A_n}(\omega) = \mathbb{1}_A(\omega)$.

(b) Let A_n be an increasing sequence of events, i.e., $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$, prove that if $A_n \to A$ then $\lim_{n\to\infty} P(A_n) = P(A)$.

(c) Let A_n be a decreasing sequence of events, i.e., $\ldots, A_3 \subseteq A_2 \subseteq A_1$, prove that if $A_n \to A$ then $\lim_{n\to\infty} P(A_n) = P(A)$.

(d) Let A_n be a sequence of events, prove that if $A_n \to A$ then $\lim_{n\to\infty} P(A_n) = P(A)$.

Problem 9. Let Ω be any set and $\mathcal{A}$ any collection of subsets of Ω. Define $\mathcal{F}(\mathcal{A})$ to be the intersection of all σ-algebras containing $\mathcal{A}$. Prove that $\mathcal{F}(\mathcal{A})$ is a σ-algebra and it is the smallest σ-algebra of subsets of Ω containing $\mathcal{A}$.

Remark This definition of the σ-algebra generated by $\mathcal{A}$ is abstract and is useful in proving things but is often useless in practice. It is often better to think $\mathcal{F}(\mathcal{A})$ as consisting of all subsets which come about as all possible unions and intersection of sets in $\mathcal{A}$.

Homework #1

#1

Prove the following defining properties of a probability space $(\Omega, \mathcal{F}, P)$

(a) If $A, B \in \mathcal{F}$ satisfy $A \cap B = \phi$ then $P(A \cup B) = P(A) + P(B)$.

(b) For all $A \in \mathcal{F}$, $P(A^c) = 1 - P(A)$

(c) For all $A, B \in \mathcal{F}$, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(d) If $A, B \in \mathcal{F}$ and $A \subseteq B$ then $P(A) \leq P(B)$.

proof

(a) $P(A \cup B) = P(A \cup B \cup \phi \cup \phi \cup \dots) = P(A) + P(B) + P(\phi) + P(\phi) + \dots = P(A) + P(B)$.

(b) $1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$

$\Rightarrow 1 - P(A) = P(A^c)$

(c) $P(A \cup B) = P((A \cap B^c) \cup (A \cap B) \cup (A^c \cap B))$

$= P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$

$= P(A \cap B^c) + P(A \cap B) - P(A \cap B) + P(A \cap B) + P(A^c \cap B) + P(A \cap B)$

$- P(A \cap B)$

$= P(A) + P(B) - P(A \cap B)$.

(d) $P(B) = P(A \cup (A^c \cap B)) = P(A) + P(A^c \cap B)$

$\Rightarrow P(B) \geq P(A)$.

$\blacksquare$

#2

Let Ω be a sample space with a finite number of elements and let $\mathcal{F}$ be set of all subsets of Ω. Prove that is we define $P: \mathcal{F} \to [0,1]$ by $P(A) = |A|/|\Omega|$ then P satisfies the three defining properties of a probability.

proof:

(i) $P(\Omega) = |\Omega|/|\Omega| = 1.$

(ii) $P(\phi) = |\phi|/|\Omega| = 0/|\Omega| = 0$

(iii) Suppose $A, B \in \Omega$ and $A \cap B = \phi$. Consequently,

$$P(A \cup B) = \frac{|A \cup B|}{|\Omega|} = \frac{|A| + |B|}{|\Omega|} = \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} = P(A) + P(B)$$

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#3

Let $\mathcal{g}_X$ be the distribution of a random variable X on some probability space $(\Omega, \mathcal{F}, P)$. Show that $\mathcal{g}_X$ satisfies the three defining properties of a probability on $\mathbb{R}$.

proof:

$$\begin{aligned} \text{(i) } \mathcal{g}_X(\mathbb{R}) &= P(X^{-1}(\mathbb{R})) \\ &= P(\{\omega : X(\omega) \in \mathbb{R}\}) \\ &= P(\Omega) \\ &= 1. \end{aligned}$$

$$\begin{aligned} \text{(ii) } \mathcal{g}_X(\phi) &= P(X^{-1}(\phi)) \\ &= P(\{\omega : X(\omega) \in \phi\}) \\ &= P(\phi) \\ &= 0 \end{aligned}$$

(iii) Suppose $A, B \in \mathbb{R}$ satisfy $A \cap B = \phi$. Therefore,

$$\begin{aligned} \rho_X(A \cup B) &= P(X^{-1}(A \cup B)) \\ &= P(\{w : X(w) \in A \cup B\}) \\ &= P(\{w : X(w) \in A\} \cup \{w : X(w) \in B\}) \\ &= P(\{w : X(w) \in A\}) + P(\{w : X(w) \in B\}) \\ &= P(X^{-1}(A)) + P(X^{-1}(B)) \\ &= \rho_X(A) + \rho_X(B). \end{aligned}$$

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#4

Consider the probability space $(\Omega, \mathcal{F}, P)$. Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of events in $\mathcal{F}$ such that $P(A_n) = 1$ for all $n \geq 1$ and let $B = \bigcap_{n=1}^{\infty} A_n$.

(a) Prove that $B \in \mathcal{F}$.

(b) Prove that $P(B) = 1$.

proof:

(a) $B = \bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{F}$.

(b) $$\begin{aligned} P(B) &= 1 - P\left(\bigcup_{n=1}^{\infty} A_n^c\right) \\ &\geq 1 - \sum_{n=1}^{\infty} P(A_n^c) \\ &= 1. \end{aligned}$$

Consequently, $P(B) = 1$.

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#5

Let X be a random variable on $(\Omega, \mathcal{F}, P)$ that is uniformly distributed on $[-1,1]$ and let $Y = X^2$.

(a) Find the CDF of Y and plot its graph.
(b) Find the PDF of Y and plot its graph.

Solution:

(a) $F(y) = P(Y \le y) = \begin{cases} 0 & \text{if } y \le 0 \\ P(0 < X^2 \le y), & 0 < y < 1 \\ 1, & y \ge 1 \end{cases}$

Now, if $0 < y < 1$

$$P(0 < X^2 \le y) = P(0 < X \le -\sqrt{y} \text{ or } -\sqrt{y} \le X < 0)$$
$$= \int_0^{\sqrt{y}} \tfrac{1}{2}\,ds + \int_{-\sqrt{y}}^{0} \tfrac{1}{2}\,ds$$
$$= \tfrac{1}{2}\sqrt{y} + \tfrac{1}{2}\sqrt{y}$$
$$= \sqrt{y}$$

Therefore,

$$F(y) = \begin{cases} 0 & \text{if } y \le 0 \\ \sqrt{y} & \text{if } 0 < y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

(b) $f_Y(y) = \frac{d}{dy} F(y) = \begin{cases} 0 & \text{if } y \le 0 \\ \frac{1}{2} y^{-1/2} & \text{if } 0 < y < 1 \\ 0 & \text{if } y \ge 1 \end{cases}$

#6.

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(a) Prove that if A and B are independent, then A^c and B are also independent.

(b) Prove that if A and B are independent, then A^c and B^c are independent.

(c) Prove that if A and B are independent then

$$P(A \cup B) = 1 - (1 - P(A))(1 - P(B))$$

proof

(a)
$$
\begin{aligned}
P(B) &= P((A^c \cup A) \cap B) \\
&= P((A^c \cap B) \cup (A \cap B)) \\
&= P(A^c \cap B) + P(A \cap B) \\
&= P(A^c \cap B) + P(A)P(B)
\end{aligned}
$$

$$\Rightarrow P(A^c \cap B) = P(B)(1 - P(A)) = P(B)P(A^c).$$

(b)
$$
\begin{aligned}
P(A^c \cap B^c) &= P((A \cup B)^c) \\
&= 1 - P(A \cup B) \\
&= 1 - P(A) - P(B) + P(A \cap B) \\
&= 1 - P(A) - P(B) + P(A)P(B) \\
&= 1 - P(B) + P(A)(P(B) - 1) \\
&= 1 - P(B) - (1 - P(B))P(A) \\
&= (1 - P(B))(1 - P(A)) \\
&= P(A^c)P(B^c).
\end{aligned}
$$

(c) $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$$= P(A) + P(B) - P(A)P(B)$$

$$= P(A)(1-P(B)) + P(B)$$

$$= 1 + P(A)(1-P(B)) + P(B) - 1$$

$$= 1 + P(A)(1-P(B)) - (1-P(B))$$

$$= 1 + (1-P(B))(P(A)-1)$$

$$= 1-(1-P(B))(1-P(A)).$$

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#7

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_n \in \mathcal{F}$. Prove that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n).$$

proof

If $A, B \in \mathcal{F}$ then $P(A\cup B) = P(A) + P(B) - P(A\cap B)$ and thus $P(A\cup B) \leq P(A) + P(B)$.
Consequently, by induction for any finite set of events we have that

$$P(C_1 \cup C_2 \cup \ldots \cup C_n) \leq \sum_{i=1}^{n} P(C_i).$$

Now, if we let $B_1 = A_1$, $B_2 = A_1 \cup A_2$, $B_3 = A_1 \cup A_2 \cup A_3, \ldots$ it follows that $B_1 \subseteq B_2 \subseteq B_3 \ldots$ and $A_n = \bigcup_{i=1}^{n} B_i$ and thus $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_n$. Therefore, by continuity of probability we have that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right) \leq \lim_{n\to\infty} \sum_{i=1}^{n} P(A_i)$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

■

#8

(a) Prove that $A_n \to A$ if and only if for all $\omega \in \Omega$,
$\lim_{n\to\infty} \mathbb{1}_{A_n}(\omega) = \mathbb{1}_A(\omega)$

(b) Let A_n be an increasing sequence of events satisfying $A_n \to A$. Prove that $\lim_{n\to\infty} P(A_n) = P(A)$.

(c) Let A_n be a decreasing sequence of events satisfing $A_n \to A$. Prove that $\lim_{n\to\infty} P(A_n) = P(A)$.

(d) Let A_n be a sequence of events. Prove that if $A_n \to A$ then $\lim_{n\to\infty} P(A_n) = P(A)$.

proof.

(a) ($\Rightarrow$) If $A_n \to A$ then for all $\omega \in A$, there exists $N(\omega)$ such that $n \geq N(\omega)$ implies $\omega \in A_n$ and thus $\mathbb{1}_{A_n}(\omega) = \mathbb{1}_A(\omega) = 1$. Therefore, $\lim_{n\to\infty} \mathbb{1}_{A_n}(\omega) = 1 = \mathbb{1}_A(\omega)$. If $\omega \notin A$ then there exists $N(\omega)$ such that $n \geq N(\omega)$ implies $\omega \notin A_n$ and thus $\mathbb{1}_{A_n}(\omega) = \mathbb{1}_A(\omega) = 0$. Therefore, $\lim_{n\to\infty} \mathbb{1}_{A_n}(\omega) = 0 = \mathbb{1}_A(\omega)$.
Consequently, for all $\omega \in \Omega$, $\lim_{n\to\infty} \mathbb{1}_{A_n}(\omega) = \mathbb{1}_A(\omega)$.

($\Leftarrow$) The argument above can be reversed to prove that $\lim_{n\to\infty} \mathbb{1}_{A_n}(\omega) = \lim_{n\to\infty} \mathbb{1}_A(\omega)$ implies $A_n \to A$

(b) By construction, $A = \bigcup_{n=1}^{\infty} A_n$. Therefore, by continuity of probability

$$P(A) = P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

(c) By construction, $A = \bigcap_{n=1}^{\infty} A_n$. Therefore, by continuity of probability

$$P(A) = P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

(d) First note that

$$A_n = (A_n \cap A) \cup (A_n \cap A^c) \text{ and } A = (A \cap A_n) \cup (A \cap A_n^c)$$

Therefore,

$$P(A_n) = P(A_n \cap A) + P(A_n \cap A^c)$$

$$P(A) = P(A \cap A_n) + P(A \cap A_n^c)$$

$$\Rightarrow |P(A_n) - P(A)| = |P(A_n \cap A^c) - P(A \cap A_n^c)| \leq P(A_n \cap A^c) + P(A \cap A_n^c)$$

$$\Rightarrow |P(A_n) - P(A)| \leq P((A_n \cap A^c) \cup (A \cap A_n^c))$$

Consequently, if we let $B_n = (A_n \cap A^c) \cup (A \cap A_n^c)$ we need to show that $\lim_{n \to \infty} P(B_n) = 0$.

Define $C_n = \bigcup_{k \geq n}^{\infty} B_k$ to be "the global difference" between the sets A_k and A for $k \geq n$. By construction C_n is a decreasing sequence of sets. Moreover, since $A_n \to A$ it follows that $C_n \to \emptyset$ and thus $\lim_{n \to \infty} P(C_n) = 0$. It also follows that since $B_n \subseteq C_n$ we have that $\lim_{n \to \infty} P(B_n) = 0$.

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