

1 Problems for all Students

Problem 1. Let X and Y be two measurable random variables on the same probability space $(\Omega, \mathcal{F}, P)$. Prove that for all $a, b \in \mathbb{R}$ the random variable $Z = aX + bY$ is also measurable.

Problem 2. Consider the probability space $(\Omega, \mathcal{F}, P)$. Given an event $B \in \mathcal{F}$ satisfying $P(B) \neq 0$, the conditional probability of A given B , denoted $P(A|B)$, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Prove that $(B, B \cap \mathcal{F}, P(\cdot|B))$ is a probability space.

Remark: The point of this problem is that if you know that B has occurred, you can now define probabilities on a new “universe” B on events in which both A and B have to occur.

Problem 3. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with density $f(x)$ and $[a, b] \subset \mathbb{R}$ for some $a, b \in \mathbb{R}$ satisfying $a < b$.

(a) If $f(x) > 0$ for all x , prove that

$$P(X < x|[a, b]) = \begin{cases} 0, & x < a \\ \frac{\int_a^x f(x)dx}{\int_a^b f(x)dx}, & a \leq x \leq b \\ 1, & x > b \end{cases}$$

(b) Explain why a reasonable definition for the conditional density of X given $[a, b]$ is the function

$$g(x) = \begin{cases} \frac{f(x)}{\int_a^b f(x)dx}, & a < x < b \\ 0, & \text{otherwise} \end{cases}.$$

Problem 4. Let Y be an exponential random variable with parameter $\lambda > 0$. Show that for any $s, t > 0$

$$P(Y > t + s | Y > s) = P(Y > t).$$

Problem 5. Let Z be a standard Gaussian random variable.

(a) Show using integration by parts that for a differentiable function g that

$$\mathbb{E}[Zg(Z)] = \mathbb{E}[g'(Z)].$$

(b) Show using induction that

$$\frac{(2j)!}{2^j j!} = (2j-1)(2j-3)(2j-5) \cdots 5 \cdot 3 \cdot 1.$$

(c) Use part (a) to show that

$$\mathbb{E}[Z^{2j}] = \frac{(2j)!}{2^j j!}.$$

2 Problems for Graduate Students

Problem 6. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with probability density function $f(x)$.

(a) Prove that if X is nonnegative, i.e., $X \geq 0$, then

$$\mathbb{E}[X] = \int_0^{\infty} P(X > x) dx.$$

(b) Prove that if X satisfies $\mathbb{E}[|X|] < \infty$, so that the expectation is well defined, then

$$\mathbb{E}[X] = \int_0^{\infty} P(X > x) dx - \int_{-\infty}^0 P(X \leq x) dx.$$

Problem 7. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with $X \geq 0$. Prove that if $\mathbb{E}[X] < \infty$, then we must have $P(X < \infty) = 1$.

Hint: $\{X = \infty\} = \bigcap_{n=1}^{\infty} \{X > n\}$.

Problem 8. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with $X \geq 0$ and suppose $\mathbb{E}[X] = 0$.

(a) Prove that $P(X = 0) = 1$.

(b) Prove that this random variable cannot have a probability density.

Homework #2

#1

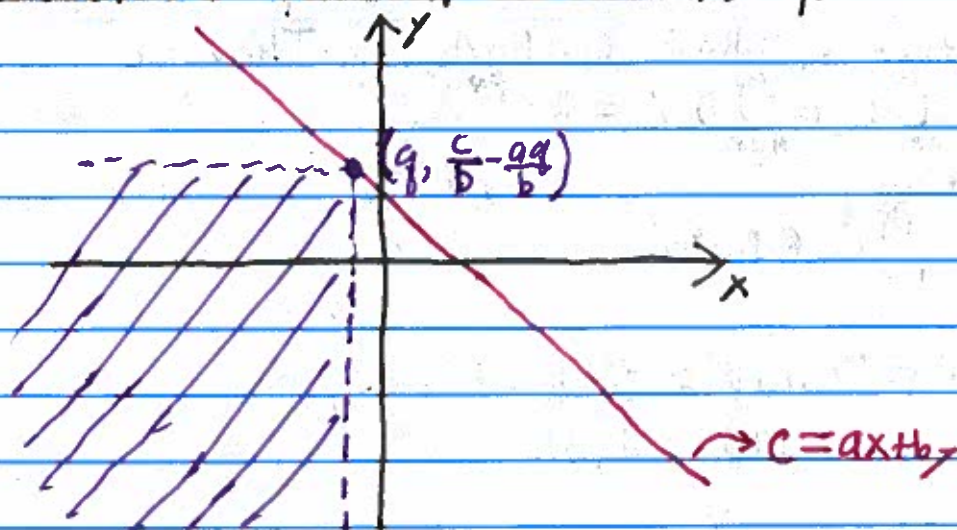
Let X, Y be two measurable random variables on the same probability space $(\Omega, \mathcal{F}, P)$. Prove that for all $a, b \in \mathbb{R}$ the random variable $Z = aX + bY$ is also measurable.

proof:

Without loss of generality assume $b \neq 0$. Therefore, for all $c \in \mathbb{R}$:

$$\{\omega : Z(\omega) < c\} = \bigcup_{q \in \mathbb{Q}} \{\omega : X(\omega) < q\} \cap \{\omega : Y(\omega) < \frac{c}{b} - \frac{aq}{b}\} \in \mathcal{F}$$

Graphically, this argument can be understood as the union of the intersection of horizontal and vertical strips!



#2

Consider the probability space $(\Omega, \mathcal{F}, P)$. Given an event $B \in \mathcal{F}$ satisfying $P(B) \neq 0$, the conditional probability of A given B , denoted $P(A|B)$, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Prove that $(B, B \cap \mathcal{F}, P(\cdot|B))$ is a probability space.

proof

1. (a) By construction $B \in B \cap \mathcal{F}$ and $\emptyset = \emptyset \cap B$ and thus $B, \emptyset \in B \cap \mathcal{F}$

(b) Suppose $C \in B \cap \mathcal{F}$. Therefore, there exists $A \in \mathcal{F}$ such that $C = B \cap A$. Consequently, $C^c = B \cap A^c \in B \cap \mathcal{F}$.

(c) Suppose $C_1, C_2, \dots \in B \cap \mathcal{F}$. Therefore there exists $A_1, A_2, \dots \in \mathcal{F}$ such that $C_1 = B \cap A_1, C_2 = B \cap A_2, \dots$. Therefore,

$$\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} B \cap A_n = B \cap \bigcup_{n=1}^{\infty} A_n$$

and thus

$$\bigcup_{n=1}^{\infty} C_n \in B \cap \mathcal{F}.$$

$$2. (a) \frac{P(B|B) = P(B \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1.$$

$$(b) \frac{P(\emptyset|B) = P(\emptyset \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = 0 = 0.$$

(c) If $A_1, A_2, \dots \in \mathcal{F}$ satisfy $A_i \cap A_j = \emptyset$ then

$$\begin{aligned} P\left(\bigcup_{i=1}^{\infty} A_i | B\right) &= \frac{P\left(\bigcup_{i=1}^{\infty} A_i \cap B\right)}{P(B)} \\ &= \frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{P(B)} \\ &= \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)} = \sum_{i=1}^{\infty} P(A_i | B). \end{aligned}$$

#3.

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with density $f(x)$ and $[a, b] \subset \mathbb{R}$ for some $a, b \in \mathbb{R}$ satisfying $a < b$.

(a) If $f(x) > 0$ for all x , prove that

$$P(X < x | [a, b]) = \begin{cases} 0, & x < a \\ \frac{\int_a^x f(x) dx}{\int_a^b f(x) dx}, & a \leq x \leq b \\ 1, & x > b. \end{cases}$$

(b) Explain why a reasonable definition for the conditional density of X given $[a, b]$ is the function

$$g(x) = \begin{cases} \frac{f(x)}{\int_a^b f(x) dx}, & a < x < b \\ 0, & \text{o.w.} \end{cases}$$

proof:

$$\begin{aligned} (a) P(X < x | [a, b]) &= \frac{P(\{\omega: X(\omega) < x\} \cap \{\omega: X(\omega) \in [a, b]\})}{P(\{\omega: X(\omega) \in [a, b]\})} \\ &= \frac{P(\{\omega: X(\omega) < x\} \cap \{\omega: X(\omega) \in [a, b]\})}{\int_a^b f(x) dx} \\ &= \frac{1}{\int_a^b f(x) dx} \begin{cases} 0, & x < a \\ \int_a^x f(x) dx, & a < x < b \\ \int_a^b f(x) dx, & b > x \end{cases} \end{aligned}$$

$$(b) \frac{d}{dx} P(X < x | [a, b]) = \begin{cases} \frac{f(x)}{\int_a^b f(x) dx}, & a < x < b \\ 0, & \text{o.w.} \end{cases}$$

#4

Let Y be an exponential random variable with parameter $\lambda > 0$. Show that for any $s, t > 0$

$$P(Y > t+s | Y > s) = P(Y > t)$$

Solution:

$$\begin{aligned} P(Y > t+s | Y > s) &= \frac{P(\{\omega: Y(\omega) > t+s\} \cap \{\omega: Y(\omega) > s\})}{P(\{\omega: Y(\omega) > s\})} \\ &= \frac{P(\{\omega: Y(\omega) > t+s\})}{P(\{\omega: Y(\omega) > s\})} \\ &= \frac{\int_{t+s}^{\infty} \lambda e^{-\lambda y} dy}{\int_s^{\infty} \lambda e^{-\lambda y} dy} \\ &= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \\ &= e^{-\lambda t} \\ &= \int_t^{\infty} \lambda e^{-\lambda y} dy \\ &= P(Y > t) \end{aligned}$$

#5

Let Z be a standard Gaussian random variable.

(a) Show using integration by parts that

$$\mathbb{E}[Zg(Z)] = \mathbb{E}[g'(Z)].$$

(b) Show using induction that

$$\frac{(2j)!}{2^j j!} = (2j-1)(2j-3)\cdots 3 \cdot 1$$

(c) Use part (a) to show that

$$\mathbb{E}[Z^{2j}] = \frac{(2j)!}{2^j j!}$$

Solution:

$$\begin{aligned} \text{(a)} \quad \mathbb{E}[Zg(Z)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \cdot g(z) e^{-z^2/2} dz \\ &= \frac{1}{\sqrt{2\pi}} \left(-e^{-z^2/2} g(z) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} g'(z) e^{-z^2/2} dz \right) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g'(z) e^{-z^2/2} dz \\ &= \mathbb{E}[g'(Z)]. \end{aligned}$$

(b) Let $P: \mathbb{N} \rightarrow \{0,1\}$ be the logical function that

$$\frac{(2j)!}{2^j j!} = (2j-1)(2j-3)\cdots 3 \cdot 1.$$

That is,

$$P(j) = \begin{cases} 1 & \text{if } \frac{(2j)!}{2^j j!} = (2j-1)(2j-3)\cdots 3 \cdot 1 \\ 0 & \text{o.w.} \end{cases}$$

Therefore,

(i) $P(1) = 1$ since

$$\frac{(2 \cdot 1)!}{2^1 \cdot 1!} = \frac{2}{2} = 1$$

(ii) Assume now that $P(n)=1$. Therefore,

$$\begin{aligned}\frac{(2 \cdot (n+1))!}{2^{n+1} \cdot (n+1)!} &= \frac{(2n+2)(2n+1)(2n)!}{2 \cdot 2^n \cdot (n+1)n!} \\ &= \frac{(2n+2)(2n+1)}{2 \cdot (n+1)} \cdot \frac{(2n)!}{2^n \cdot n!} \\ &= (2n+1) \cdot (2n-1)(2n-3) \cdots 3 \cdot 1\end{aligned}$$

and thus $P(n+1)=1$.

By item (i) and (ii) and the principle of mathematical induction it follows that for all $j \in \mathbb{N}$

$$\frac{(2j)!}{2^j \cdot j!} = (2j-1)(2j-3) \cdots 3 \cdot 1$$

$$\begin{aligned}c) E[Z^{2j}] &= E[Z \cdot Z^{2j-1}] \\ &= E[(2j-1)Z^{2j-2}] \\ &= (2j-1)E[Z \cdot Z^{2j-3}] \\ &= (2j-1)(2j-3)E[Z^{2j-4}] \\ &\vdots \\ &= (2j-1)(2j-3) \cdots 3 \cdot E[1] \\ &= (2j-1)(2j-3) \cdots 3 \cdot 1 \\ &= \frac{(2j)!}{2^j \cdot j!}\end{aligned}$$

#6

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with probability density $f(x)$.

(a) Prove that if X is nonnegative then

$$E[X] = \int_0^{\infty} P(X > x) dx.$$

(b) Prove that if X satisfies $E[|X|] < \infty$ then

$$E[X] = \int_0^{\infty} P(X > x) dx - \int_{-\infty}^0 P(X \leq x) dx$$

proof:

$$\begin{aligned} (a) E[X] &= \int_0^{\infty} x f(x) dx \quad \text{(Assuming } f \text{ decays rapidly enough)} \\ &= -x \int_x^{\infty} f(s) ds \Big|_0^{\infty} + \int_0^{\infty} \int_x^{\infty} f(s) ds dx \\ &= \int_0^{\infty} \int_x^{\infty} f(s) ds dx \\ &= \int_0^{\infty} P(X > x) dx. \end{aligned}$$

$$(b) X = 1_{\{X > 0\}} X + 1_{\{X \leq 0\}} X$$

$$\begin{aligned} \Rightarrow E[X] &= E[1_{\{X > 0\}} X] + E[1_{\{X \leq 0\}} X] \\ &= E[1_{\{X > 0\}} X] - E[1_{\{X \leq 0\}} (-X)] \\ &= \int_0^{\infty} P(1_{\{X > 0\}} X > x) dx - \int_0^{\infty} P(1_{\{X \leq 0\}} (-X) > x) dx \\ &= \int_0^{\infty} P(X > x) dx - \int_0^{\infty} P(1_{\{X \leq 0\}} X < -x) dx \\ &= \int_0^{\infty} P(X > x) dx - \int_0^{\infty} P(X < -x) dx \\ &= \int_0^{\infty} P(X > x) dx + \int_0^{\infty} P(X < u) du \\ &= \int_0^{\infty} P(X > x) dx - \int_0^0 P(X < u) du \\ &= \int_0^{\infty} P(X > x) dx - \int_{-\infty}^0 P(X < x) dx \end{aligned}$$

#7

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with $X \geq 0$. Prove that if $E[X] < \infty$, then we must have $P(X < \infty) = 1$.

proof:

By continuity of probability and Markov's inequality we have that

$$P(\{X = \infty\}) = P\left(\bigcap_{n=1}^{\infty} \{X > n\}\right) = \lim_{n \rightarrow \infty} P(X > n) \leq \lim_{n \rightarrow \infty} \frac{1}{n} E[X] = 0.$$

Consequently, $P(\{X = \infty\}) = 0$ and thus

$$P(X < \infty) = 1 - P(\{X = \infty\}) = 1.$$

#8

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with $X \geq 0$ and suppose $E[X] = 0$.

(a) Prove that $P(X = 0) = 1$.

(b) Prove that this random variable cannot have a probability density.

Solution:

(a) By continuity of probability and Markov's inequality we have that

$$P(\{X > 0\}) = P\left(\bigcup_{n=1}^{\infty} \{X > \frac{1}{n}\}\right) = \lim_{n \rightarrow \infty} P(X > \frac{1}{n}) \leq \lim_{n \rightarrow \infty} n E[X] = 0.$$

Consequently,

$$P(X = 0) = 1 - P(\{X > 0\}) = 1.$$

(b) Suppose X has a density. Consequently, the function $F: [0, \infty) \rightarrow \mathbb{R}$ defined by $F(x) = P(0 \leq X \leq x)$ is continuous and satisfies

$$1 = F(x) = \int_0^x f(s) ds.$$

and thus $1 = \lim_{x \rightarrow \infty} F(x) = \int_0^{\infty} f(s) ds = 0$ which is a contradiction.