

1 Problems for all Students

Problem 1. Let $\vec{X} = (X, Y)$ be a random vector with joint density

$$f(x, y) = \begin{cases} 6x^2y & 0 \leq x \leq y \text{ and } x + y \leq 2 \\ 0 & \text{elsewhere} \end{cases}.$$

- (a) Find the probability density functions of X and Y .
- (b) Are X and Y independent random variables?

Problem 2. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with an exponential distribution with parameter $\lambda > 0$.

- (a) Show that the moment generating function of X is given by

$$\phi(t) = \mathbb{E}[e^{tX}] = \frac{\lambda}{\lambda - t} \text{ for } t < \lambda.$$

- (b) Use $\phi(t)$ to compute the expectation and variance of X .

Problem 3. Let $X_1, \dots, X_n$ be n independent and identically distributed exponential random variables with parameter $\lambda > 0$ and let $Z = X_1 + \dots + X_n$.

- (a) Find the moment generating function of Z .
- (b) A random variable Y is said to have a gamma distribution with parameter $\lambda > 0$ if it has the following probability density

$$f(y) = \begin{cases} \frac{\lambda^n}{(n-1)!} y^{n-1} e^{-\lambda y}, & y \geq 0 \\ 0, & y < 0 \end{cases},$$

where $n \in \mathbb{N}$. By finding the moment generating function of Y , prove that the sum of n independent and identically distributed exponential random variables is a gamma distribution.

Problem 4. Let Z be a standard Gaussian random variable.

- (a) Show that for $\sigma > 0$ and $\mu \in \mathbb{R}$ the random variable $X = \sigma Z + \mu$ is also a Gaussian random variable with mean μ and variance σ^2 .
- (b) Show that the moment generating function of a Gaussian random variable X with mean μ and variance σ^2 is given by

$$\phi(t) = \mathbb{E}[e^{tX}] = e^{t\mu + t^2\sigma^2/2}.$$

Problem 5. Let X_1, X_2 be two independent but not necessarily identically distributed Gaussian random variables. By computing its moment generating function, prove that $Z = X_1 + X_2$ is also a Gaussian random variable.

Homework #3

#1.

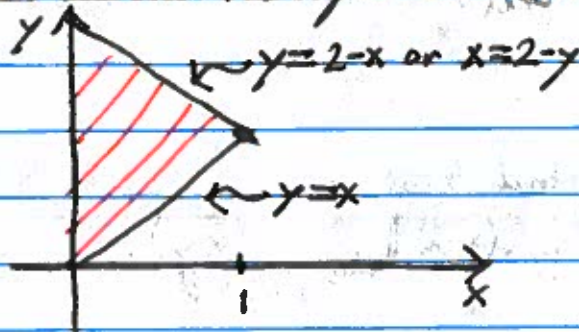
Let $\vec{X} = (X, Y)$ be a random vector with joint density

$$f(x, y) = \begin{cases} 6x^2y, & 0 \leq x \leq y \text{ and } x+y \leq 2 \\ 0, & \text{o.w.} \end{cases}$$

- (a) Find the probability density functions of X and Y .
(b) Are X and Y independent random variables?

Solution:

We can sketch the region on which f is supported



$$(a) P(X < a) = \begin{cases} 0, & a < 0 \\ \int_0^a \int_x^{2-x} 6x^2y \, dy \, dx, & 0 \leq a \leq 1 \\ 1, & a > 1 \end{cases}$$

Therefore, the density of X is given by

$$\begin{aligned} f(a) &= \frac{d}{da} P(X < a) = \int_0^{2-a} 6a^2y \, dy, \quad 0 \leq a \leq 1 \\ &= \begin{cases} 0, & \text{o.w.} \\ 3a^2((2-a)^2 - a^2), & 0 \leq a \leq 1 \\ 0, & \text{o.w.} \end{cases} \\ &= \begin{cases} 12a^2 - 12a^3, & 0 \leq a \leq 1 \\ 0, & \text{o.w.} \end{cases} \end{aligned}$$

$$\Rightarrow X \sim \begin{cases} 12x^2 - 12x^3, & 0 \leq x \leq 1 \\ 0, & \text{o.w.} \end{cases}$$

$$P(Y \leq b) = \begin{cases} 0, & b < 0 \\ \int_0^b \int_0^x 6x^2 y \, dx \, dy, & 0 \leq b \leq 1 \\ \int_0^1 \int_0^x 6x^2 y \, dx \, dy + \int_1^b \int_0^{2-y} 6x^2 y \, dx \, dy, & 1 \leq b \leq 2 \\ 1, & b \geq 2 \end{cases}$$

Therefore, the density of Y is given by

$$g(b) = \frac{dP(Y \leq b)}{db} = \begin{cases} \int_0^b 6x^2 \, dx, & 0 \leq b \leq 1 \\ \int_0^{2-b} 6x^2 \, dx, & 1 \leq b \leq 2 \\ 0, & \text{o.w.} \end{cases}$$

$$= \begin{cases} 2b^3, & 0 \leq b \leq 1 \\ 2b(2-b)^2, & 1 \leq b \leq 2 \\ 0, & \text{o.w.} \end{cases}$$

(b) X and Y are not independent since

$$f(x, y) = 6x^2 y \mathbb{1}_{\{0 \leq x \leq y\}}(x, y) \mathbb{1}_{\{x+y \leq 2\}}(x, y)$$

which is not separable.

#2

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with an exponential distribution with parameter $\lambda > 0$.

(a) Show that the moment generating function of X is given by

$$\phi(t) = \mathbb{E}[e^{tX}] = \frac{\lambda}{\lambda - t} \text{ for } t < \lambda.$$

(b) Use $\phi(t)$ to compute the expectation and variance of X .

Solution:

$$(a) \phi(t) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \frac{\lambda}{\lambda - t} \text{ if } t < \lambda.$$

$$(b) \phi'(t) = \frac{\lambda}{(\lambda-t)^2} \Rightarrow E[X] = \phi'(0) = 1/\lambda$$

$$\phi''(t) = \frac{2\lambda}{(\lambda-t)^3} \Rightarrow E[X^2] = \phi''(0) = 2/\lambda^2$$

Consequently,

$$\sigma^2 = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2.$$

#3

Let $X_1, \dots, X_n$ be n -independent and identically distributed exponential random variables with parameter $\lambda > 0$ and let $Z = X_1 + \dots + X_n$.

(a) Find the moment generating function of Z .

(b) Let Y be a gamma distribution with parameter $\lambda > 0$.

By finding the moment generating function of Y , prove that the sum of n independent and identically distributed exponential random variables is a gamma distribution.

Solution:

$$(a) \phi(t) = E[e^{tZ}] = E[e^{t(X_1 + \dots + X_n)}] = E[e^{tX_1} e^{tX_2} \dots e^{tX_n}]$$

$$\Rightarrow \phi(t) = E[e^{tX_1}] \dots E[e^{tX_n}] = \frac{\lambda}{\lambda-t} \dots \frac{\lambda}{\lambda-t} = \frac{\lambda^n}{(\lambda-t)^n}$$

(b) For the gamma distribution

$$\begin{aligned} \phi(t) &= E[e^{tY}] = \int_0^\infty \frac{\lambda^n}{(n-1)!} y^{n-1} e^{-\lambda y} e^{ty} dy \\ &= \int_0^\infty \frac{\lambda^n}{(n-1)!} \frac{(\lambda-t)^n}{(\lambda-t)^n} y^{n-1} e^{-(\lambda-t)y} dy \\ &= \frac{\lambda^n}{(\lambda-t)^n} \int_0^\infty \frac{(\lambda-t)^n}{(n-1)!} y^{n-1} e^{-(\lambda-t)y} dy \\ &= \lambda^n / (\lambda-t)^n. \end{aligned}$$

#4

Let Z be a standard Gaussian random variable

(a) Show that for $\sigma > 0$ and $\mu \in \mathbb{R}$ the random variable $X = \sigma Z + \mu$ is also a Gaussian random variable with mean μ and variance σ^2 .

(b) Show that the moment generating function of a Gaussian random variable with mean μ and variance σ^2 is given by

$$\phi(t) = \mathbb{E}[e^{tX}]$$

Solution:

$$(a) P(X \leq x) = P(\sigma Z + \mu \leq x) = P(Z \leq \frac{x - \mu}{\sigma}) = \int_{-\infty}^{\frac{x - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$
$$\Rightarrow f(x) = \frac{d}{dx} P(X \leq x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

$$(b) \phi(t) = \mathbb{E}[e^{tX}] = \mathbb{E}[e^{t(\sigma Z + \mu)}] = e^{t\mu} \mathbb{E}[e^{t\sigma Z}]$$

$$\begin{aligned} \Rightarrow \phi(t) &= e^{t\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z} e^{-z^2/2} dz \\ &= e^{t\mu} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z + t^2\sigma^2)} e^{\frac{1}{2}t^2\sigma^2} dz \\ &= e^{t\mu} e^{\frac{1}{2}t^2\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z - t\sigma)^2} \frac{1}{\sqrt{2\pi}} dz \\ &= e^{t\mu} e^{\frac{1}{2}t^2\sigma^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} \frac{1}{\sqrt{2\pi}} du \\ &= e^{t\mu + \frac{1}{2}t^2\sigma^2} \end{aligned}$$

#5

Let X_1, X_2 be two independent Gaussian random variables. By computing the moment generating function, prove that $Z = X_1 + X_2$ is also a Gaussian random variable.

Solution:

Let μ_1, μ_2 and σ_1, σ_2 denote the mean and variance of X_1, X_2 respectively. Therefore, for $Z = X_1 + X_2$ we have that

$$\begin{aligned}
\mathbb{E}[e^{tZ}] &= \mathbb{E}[e^{tX_1 + tX_2}] \\
&= \mathbb{E}[e^{tX_1} e^{tX_2}] \\
&= \mathbb{E}[e^{tX_1}] \mathbb{E}[e^{tX_2}] \\
&= e^{t\mu_1 + t^2\sigma_1^2/2} e^{t\mu_2 + t^2\sigma_2^2/2} \\
&= e^{t(\mu_1 + \mu_2)} e^{t^2(\sigma_1^2 + \sigma_2^2)/2}
\end{aligned}$$

Consequently, Z_1 is a Gaussian random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. ■

