Homework #1, Due Date: 09/04/26
MTH 383/683: Fall 2026
Instructor: John Gemmer

1 Problems for all Students

Problem 1. Prove the following defining properties of a probability space $(\Omega, \mathcal{F}, P)$.

(a) If $A, B \in \mathcal{F}$ satisfy $A \cap B = \emptyset$ then $P(A \cup B) = P(A) + P(B)$,

(b) For all $A \in \mathcal{F}$, $P(A^c) = 1 - P(A)$,

(c) For all $A, B \in \mathcal{F}$, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$,

(d) If $A, B \in \mathcal{F}$ and $A \subseteq B$ then $P(A) \leq P(B)$.

Problem 2. Let Ω be a sample space with a finite number of elements and let $\mathcal{F}$ be the set of all subsets of Ω. Prove that if we define $P : \mathcal{F} \mapsto [0, 1]$ by $P(A) = \frac{|A|}{|\Omega|}$ then P satisfies the three defining properties of a probability. Note, in this problem $|\cdot|$ denotes the cardinality of a set.

Problem 3. Let ρ_X be the distribution of a random variable X on some probability space $(\Omega, \mathcal{F}, P)$. Show that ρ_X satisfies the three defining properties of a probability on $\mathbb{R}$.

Problem 4. Consider the probability space $(\Omega, \mathcal{F}, P)$. Let $A_1, A_2, A_3, \ldots$ be an infinite sequence of events in $\mathcal{F}$ such that $P(A_n) = 1$ for all $n \geq 1$ and let $B = \bigcap_{n=1}^{\infty} A_n$.

(a) Prove that $B \in \mathcal{F}$,

(b) Prove that $P(B) = 1$.

Problem 5. Let X be a random variable on $(\Omega, \mathcal{F}, P)$ that is uniformly distributed on $[-1, 1]$ and let $Y = X^2$.

(a) Find the CDF of Y and plot its graph.

(b) Find the PDF of Y and plot its graph.

Problem 6. Let $(\Omega, \mathcal{F}, P)$ be some probability space. Recall that two events $A, B \in \mathcal{F}$ are independent if $P(A \cap B) = P(A)P(B)$.

(a) Prove that if A and B are independent, then A^c and B are also independent.

(b) Prove that if A and B are independent, then A^c and B^c are also independent.

(c) Prove that if A and B are independent, then

$$P(A \cup B) = 1 - (1 - P(A))(1 - P(B)).$$

2 Problems for Graduate Students

Problem 7. Hint: You should work through the proof of Lemma 1.4 in the text before doing this problem. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_n \in \mathcal{F}$ be a sequence of events. Prove that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n).$$

Problem 8. Hint: You should work through the proof of Lemma 1.4 in the text before doing this problem. Let $(\Omega, \mathcal{F}, P)$ be a probability space. We say a sequence of events A_n converges to A as $n \to \infty$, written as $A_n \to A$, if for all $\omega \in A$ there exists $N(\omega) \in \mathbb{N}$ such that for all $n \geq N(\omega)$, $\omega \in A_n$ if and only if $\omega \in A$.

(a) Prove that $A_n \to A$ if and only if for all $\omega \in \Omega$, $\lim\limits_{n\to\infty} \mathbb{1}_{A_n}(\omega) = \mathbb{1}_A(\omega)$.

(b) Let A_n be an increasing sequence of events, i.e., $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$, prove that if $A_n \to A$ then $\lim\limits_{n\to\infty} P(A_n) = P(A)$.

(c) Let A_n be a decreasing sequence of events, i.e., $\ldots, A_3 \subseteq A_2 \subseteq A_1$, prove that if $A_n \to A$ then $\lim\limits_{n\to\infty} P(A_n) = P(A)$.

(d) Let A_n be a sequence of events, prove that if $A_n \to A$ then $\lim\limits_{n\to\infty} P(A_n) = P(A)$.

Problem 9. Let Ω be any set and $\mathcal{A}$ any collection of subsets of Ω. Define $\mathcal{F}(\mathcal{A})$ to be the intersection of all σ-algebras containing $\mathcal{A}$. Prove that $\mathcal{F}(\mathcal{A})$ is a σ-algebra and it is the smallest σ-algebra of subsets of Ω containing $\mathcal{A}$.

Remark This definition of the σ-algebra generated by $\mathcal{A}$ is abstract and is useful in proving things but is often useless in practice. It is often better to think $\mathcal{F}(\mathcal{A})$ as consisting of all subsets which come about as all possible unions and intersection of sets in $\mathcal{A}$.