Homework #2, Due Date: 09/11/26
MTH 383/683: Fall 2026
Instructor: John Gemmer

1 Problems for all Students

Problem 1. Let X and Y be two measurable random variables on the same probability space $(\Omega, \mathcal{F}, P)$. Prove that for all $a, b \in \mathbb{R}$ the random variable $Z = aX + bY$ is also measurable.

Problem 2. Consider the probability space $(\Omega, \mathcal{F}, P)$. Given an event $B \in \mathcal{F}$ satisfying $P(B) \neq 0$, the conditional probability of A given B, denoted $P(A|B)$, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Prove that $(B, B \cap \mathcal{F}, P(\cdot|B))$ is a probability space.

Remark: The point of this problem is that if you know that B has occurred, you can now define probabilities on a new "universe" B on events in which both A and B have to occur.

Problem 3. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with density $f(x)$ and $[a, b] \subset \mathbb{R}$ for some $a, b \in \mathbb{R}$ satisfying $a < b$.

(a) If $f(x) > 0$ for all x, prove that

$$P(X < x|[a, b]) = \begin{cases} 0, & x < a \\ \dfrac{\int_a^x f(x)dx}{\int_a^b f(x)dx}, & a \leq x \leq b \\ 1, & x > b \end{cases}$$

(b) Explain why a reasonable definition for the conditional density of X given $[a, b]$ is the function

$$g(x) = \begin{cases} \dfrac{f(x)}{\int_a^b f(x)dx}, & a < x < b \\ 0, & \text{otherwise} \end{cases}.$$

Problem 4. Let Y be an exponential random variable with parameter $\lambda > 0$. Show that for any $s, t > 0$

$$P(Y > t + s|Y > s) = P(Y > t).$$

Problem 5. Let Z be a standard Gaussian random variable.

(a) Show using integration by parts that for a differentiable function g that

$$\mathbb{E}[Zg(Z)] = \mathbb{E}[g'(Z)].$$

(b) Show using induction that

$$\frac{(2j)!}{2^j j!} = (2j-1)(2j-3)(2j-5) \cdot \ldots \cdot 5 \cdot 3 \cdot 1.$$

(c) Use part (a) to show that

$$\mathbb{E}[Z^{2j}] = \frac{(2j)!}{2^j j!}.$$

2 Problems for Graduate Students

Problem 6. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with probability density function $f(x)$.

(a) Prove that if X is nonnegative, i.e., $X \geq 0$, then

$$\mathbb{E}[X] = \int_0^\infty P(X > x)\, dx.$$

(b) Prove that if X satisfies $\mathbb{E}[|X|] < \infty$, so that the expectation is well defined, then

$$\mathbb{E}[X] = \int_0^\infty P(X > x)\, dx - \int_{-\infty}^0 P(X \leq x)\, dx.$$

Problem 7. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with $X \geq 0$. Prove that if $\mathbb{E}[X] < \infty$, then we must have $P(X < \infty) = 1$.

Hint: $\{X = \infty\} = \bigcap_{n=1}^\infty \{X > n\}$.

Problem 8. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with $X \geq 0$ and suppose $\mathbb{E}[X] = 0$.

(a) Prove that $P(X = 0) = 1$.

(b) Prove that this random variable cannot have a probability density.