

1 Problems for all Students

Problem 1. Let $\vec{X} = (X, Y)$ be a random vector with joint density

$$f(x, y) = \begin{cases} 6x^2y & 0 \leq x \leq y \text{ and } x + y \leq 2 \\ 0 & \text{elsewhere} \end{cases}.$$

- (a) Find the probability density functions of X and Y .
- (b) Are X and Y independent random variables?

Problem 2. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ with an exponential distribution with parameter $\lambda > 0$.

- (a) Show that the moment generating function of X is given by

$$\phi(t) = \mathbb{E}[e^{tX}] = \frac{\lambda}{\lambda - t} \text{ for } t < \lambda.$$

- (b) Use $\phi(t)$ to compute the expectation and variance of X .

Problem 3. Let $X_1, \dots, X_n$ be n independent and identically distributed exponential random variables with parameter $\lambda > 0$ and let $Z = X_1 + \dots + X_n$.

- (a) Find the moment generating function of Z .
- (b) A random variable Y is said to have a gamma distribution with parameter $\lambda > 0$ if it has the following probability density

$$f(y) = \begin{cases} \frac{\lambda^n}{(n-1)!} y^{n-1} e^{-\lambda y}, & y \geq 0 \\ 0, & y < 0 \end{cases},$$

where $n \in \mathbb{N}$. By finding the moment generating function of Y , prove that the sum of n independent and identically distributed exponential random variables is a gamma distribution.

Problem 4. Let Z be a standard Gaussian random variable.

- (a) Show that for $\sigma > 0$ and $\mu \in \mathbb{R}$ the random variable $X = \sigma Z + \mu$ is also a Gaussian random variable with mean μ and variance σ^2 .
- (b) Show that the moment generating function of a Gaussian random variable X with mean μ and variance σ^2 is given by

$$\phi(t) = \mathbb{E}[e^{tX}] = e^{t\mu + t^2\sigma^2/2}.$$

Problem 5. Let X_1, X_2 be two independent but not necessarily identically distributed Gaussian random variables. By computing its moment generating function, prove that $Z = X_1 + X_2$ is also a Gaussian random variable.