

1 Problems for all Students

Problem 1. Let X be a standard Gaussian random variable. Show that $\text{Cov}(X, X^2) = 0$, yet X and X^2 are not independent.

Problem 2. Let X be a standard Gaussian random variable and define the random variable Y by

$$Y = \begin{cases} X & \text{if } |X| \leq 1, \\ -X & \text{if } |X| > 1. \end{cases}$$

- (a) Show that Y is also a standard Gaussian random variable.
- (b) Argue that $X + Y$ is not a Gaussian random variable.
- (c) Explain why the random vector $\vec{X} = (X, Y)$ is not a Gaussian random vector.

Remark: This is an important example to keep in mind. It explains why we cannot define a Gaussian random vector by simply stating that each of its components is Gaussian.

Problem 3. Let C be the covariance matrix of the random vector $(X_1, \dots, X_n)$ (not necessarily Gaussian). Show that C is always positive semidefinite, i.e.,

$$\sum_{i=1}^n \sum_{j=1}^n a_i a_j C_{ij} \geq 0,$$

for all $a_1, \dots, a_n \in \mathbb{R}$. **Hint:** Write the left side as the variance of some random variable.

Problem 4. Let $\vec{X} = (X_1, \dots, X_n)$ be an n -dimensional Gaussian vector and M an $m \times n$ matrix.

- (a) Show that $\vec{Y} = M\vec{X}$ is also a Gaussian vector.
- (b) If the covariance matrix of $\vec{X}$ is C , write the covariance matrix of $\vec{Y}$ in terms of M and C .

Problem 5. Let $\vec{X} = (X, Y)$ be a Gaussian random vector with mean 0 and covariance matrix

$$C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

for $\rho \in (-1, 1)$. Write (X, Y) as a linear combination of IID standard Gaussian random variables. Write down the PDF of the Gaussian vector.

Problem 6. Consider the Gaussian vector $\vec{X} = (X_1, X_2, X_3)$ with mean 0 and covariance matrix

$$C = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

If $\vec{Z} = (Z_1, Z_2, Z_3)$, with Z_1, Z_2, Z_3 IID standard Gaussian random variables, find a matrix A such that $\vec{X} = A\vec{Z}$.

2 Problems for Graduate Students

Problem 7. Let $\vec{X} = (X_1, \dots, X_n)$ be a random vector with covariance matrix C .

- (a) Show that C is symmetric, i.e., $C^T = C$.
- (b) Let λ_1 and λ_2 be distinct eigenvalues of C , with corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$. Show that $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1^T \vec{v}_2 = 0$. **Hint:** Compute $\vec{v}_1^T C \vec{v}_2$ in two different ways.
- (c) Suppose λ is an eigenvalue of C with corresponding eigenvector $\vec{v} \neq \vec{0}$. Using the fact that C is positive semidefinite, show that $\lambda \geq 0$.
- (d) Recall that a real-valued $n \times n$ matrix Q is called *unitary* if its columns form an orthonormal basis of $\mathbb{R}^n$. Show that if Q is unitary, then $Q^{-1} = Q^T$.
- (e) Explain why for any covariance matrix C there exists a unitary matrix Q and a diagonal matrix $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ such that $C = Q\Lambda Q^T$.
- (f) Show that there exists a matrix A such that $C = AA^T$.

Problem 8. Let $\vec{X} = (X_1, \dots, X_n)$ be a Gaussian random vector with mean 0 and covariance matrix C . If $\vec{Z} = (Z_1, \dots, Z_n)$, where Z_i are IID standard Gaussian random variables, show that there exists a matrix A such that $\vec{X} = A\vec{Z}$.