Lecture #1: Probability Spaces

Example:

Flip a coin, what happens?
- Fall on heads, tails
- Fall on edge
- Roll away
- Explode
- Get stolen

$\vdots$

There is a lot of uncertainty, dictated by many things.
We build a mathematical universe Ω in which the only outcomes are H, T, i.e.,

$$\Omega = \{H, T\}$$

We define a probability by

$$\underbrace{P(\phi)=0}_{\text{probability nothing happens}},\ \underbrace{P(H)=\tfrac{1}{2}}_{\text{probability heads}},\ \underbrace{P(T)=\tfrac{1}{2}}_{\text{probability tails}},\ \underbrace{P(\Omega)=1}_{\text{probability something happens}}$$

- Ω is called the sample space.
- A subset of Ω is called an event.

A σ-field or a measurable collection of events is a set of subsets $\mathcal{F}$ of Ω satisfying

1. $\phi, \Omega \in \mathcal{F}$ (Nothing or something happens is measurable)
2. If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$ (If A can be measured then A not happening can be measured).
3. If $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ (If all A_i are measurable then A_1 or A_2 or $A_3 \ldots$ can be measured.

Example:

$\Omega = \{R, I, D\}$ republican, ind., democrat

Possible measurable collections:

$\mathcal{F}_1 = \{\phi, \{R, I, D\}\}$ (measure if one of three major affiliations)

$\mathcal{F}_2 = \{\phi, \Omega, \{R\}, \{I D\}\}$ (measure if republican).

$\mathcal{F}_3 = \{\phi, \Omega, \{R, D\}, \{I\}\}$ (measure if republican or democrat)

$\mathcal{F}_4 = \{\phi, \Omega, \{R\}, \{D\}, \{I\}, \{D, I\}, \{R, I\}, \{R, D\}\}$

$\vdots$

All of these different σ-algebras encode different information.

$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_4$ (Example of a filtration)

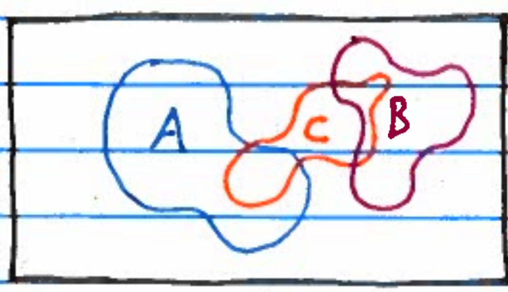
Definition: A probability P is a function $P: \mathcal{F} \to [0,1]$ satisfying

(i) $P(\phi) = 0$, $P(\Omega) = 1$

(ii) If $A_1, A_2, \ldots$ is an infinite sequence of mutually exclusive events ($A_i \cap A_j = \phi$ if $i \neq j$) then

$$\underbrace{P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots}_{\text{replicates properties of area}}$$

Example:



$P(A \cup B) = P(A) + P(B)$

Definition - A probability space $(\Omega, \mathcal{F}, P)$ is a triple

- Ω is the sample space
- $\mathcal{F}$ is a σ-field
- $P: \mathcal{F} \to [0,1]$ is a probability function.

Why Care?

Take a circle of radius 2 inches in the plane and choose a chord of this circle at random. What is the probability this chord intersects a concentric circle of radius 1 inch?

Solution 1:

Any chord is determined by the location of its midpoint

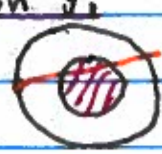
$$\Rightarrow \text{probability} = \frac{\text{area of inner circle}}{\text{area of outer circle}} = \frac{\pi}{4\pi} = \frac{1}{4}$$

Solution 2:

By symmetry under rotation we may assume chord is vertical

$$\Rightarrow \text{probability} = \frac{\text{diameter of inner circle}}{\text{diameter of outer circle}} = \frac{2}{4} = \frac{1}{2}$$

Solution 3:



$$\text{probability} = \frac{2\pi/6}{\pi} = \frac{1}{3}$$

Proposition -

1. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$
2. $P(A^c) = 1 - P(A)$
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
4. If $A \subseteq B$ then $P(A) \leq P(B)$.

Proof:

1. If $A, B \in \mathcal{F}$ then $A^c, B^c \in \mathcal{F}$ and thus $A \cap B = (A^c \cup B^c)^c \in \mathcal{F}$.
2. Since $A \dot{\cup} A^c = \Omega$ and $A \cap A^c = \emptyset$ it follows that
$$1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$$
$$\Rightarrow P(A^c) = 1 - P(A).$$
3. $$\begin{aligned} P(A \cup B) &= P((A \cap B^c) \cup (A \cap B) \cup (A^c \cap B)) \\ &= P(A \cap B^c) + P(A \cap B) + P(A^c \cap B) \\ &= \underbrace{P(A \cap B^c) + P(A \cap B)}_{P(A)} - P(A \cap B) + \underbrace{P(A \cap B) + P(A^c \cap B)}_{P(B)} + P(A \cap B) \\ &\quad - P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$
4. $$\begin{aligned} P(B) &= P(A \cup (A^c \cap B)) \\ &= P(A) + P(A^c \cap B) \end{aligned}$$
$$\Rightarrow P(B) \geq P(A)$$