Lecture #2: Random Variables

Definition- A random variable is a function X from Ω to $\mathbb{R}$, i.e., $X: \Omega \to \mathbb{R}$, such that $X^{-1}((-\infty, a)) \in \mathcal{F}$ for all $a \in \mathbb{R}$.

Example:

Flip a coin twice.

$\Rightarrow \Omega = \{(H,H), (H,T), (T,H), (T,T)\}$

$\mathcal{F}$ is generated by Ω, i.e., is the set of all subsets.

Principle of indifference:

$$P((H,H)) = 1/4,\ P((H,T)) = 1/4,\ P((T,H)) = 1/4,\ P((T,T)) = 1/4.$$

Define

$X(\omega)$ = sum of heads that appear

Notation:

$$
\begin{aligned}
- P(X=2) &= P(\{\omega : X(\omega) = 2\}) \\
&= P(X^{-1}(2)) \\
&= P((H,H)) \\
&= 1/4 \\
- P(X=1) &= P(\{\omega : X(\omega) = 1\}) \\
&= P(X^{-1}(1)) \\
&= P((H,T) \cup (T,H)) \\
&= P((H,T)) + P((T,H)) \\
&= 1/4 + 1/4 = 1/2 \\
- P(X=0) &= P(\{\omega : X(\omega) = 0) \\
&= P((0,0)) \\
&= 1/4
\end{aligned}
$$

Defines new probability on a new sample space $\Omega' = \{0,1,2\}$. with $\mathcal{F}'$ the power set.

- $P(X=-1)=P(\{\omega: X(\omega)=-1\})$
$$=P(\emptyset)$$
$$=0$$
- $P(X\geq 1)=P(\{\omega: X(\omega)\geq 1\})$
$$=P((H,T)\cup(T,H)\cup(H,H))$$
$$=P((H,T))+P((T,H))+P((H,H))$$
$$=3/4.$$

Definition - The distribution of a random variable X, denoted ρ_X is a function on subsets $A\subseteq\mathbb{R}$ such that
$$\rho_X(A)=P(X\in A)$$
$$=P(\{\omega\in\Omega: X(\omega)\in A\})$$
$$=P(X^{-1}(A))$$

* Implicitly, we are assuming $X^{-1}(A)\in\mathcal{F}$. Such functions are called measurable.

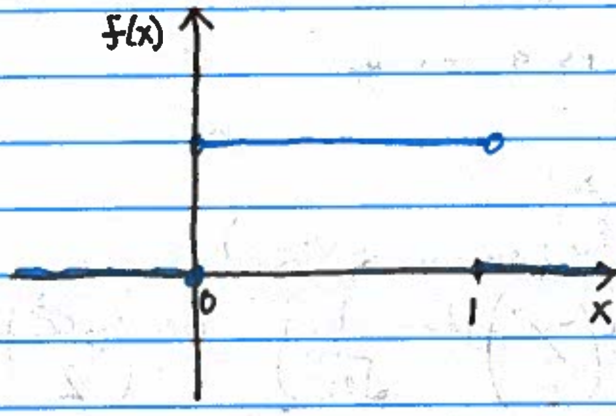
Definition - The cummulative distribution function (CDF) is defined by $F_X:\mathbb{R}\to[0,1]$ defined by
$$F_X(x)=P(X\leq x)$$
$$\rightarrow \rho_X([a,b])=F_X(b)-F_X(a)$$
If F_X is differentiable, there exists a probability density function (PDF) $f(x)$ such that
$$\rho_X((a,b))=F_X(b)-F_X(a)$$
$$=\int_a^b F'(x)\,dx$$
$$=\int_a^b f(x)\,dx$$

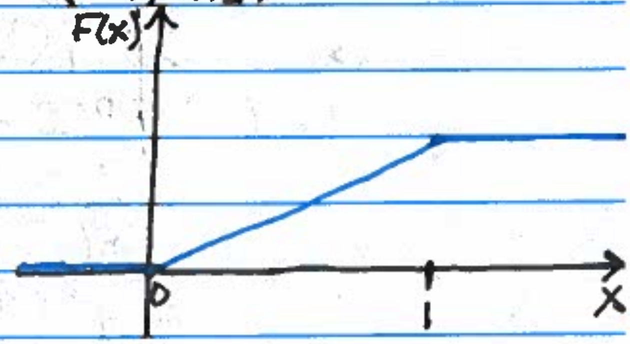
* The key point is that a distribution defines a probability on $\mathbb{R}$.

Examples

1. Uniform Distribution

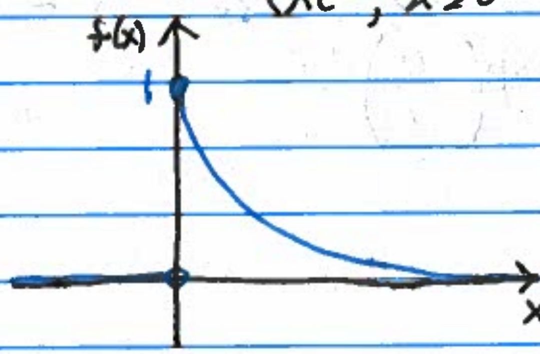
$$f(x) = \begin{cases} 1, & 0<x<1 \\ 0, & \text{o.w.} \end{cases} \Rightarrow F(x) = \begin{cases} 0, & x\le 0 \\ x, & 0\le x\le 1 \\ 1, & x\ge 1 \end{cases}$$

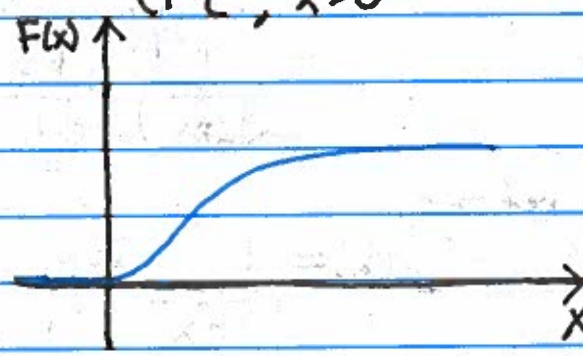




2. Exponential Distribution

$$f(x) = \begin{cases} 0, & x\le 0 \\ \lambda e^{-\lambda x}, & x>0 \end{cases} \Rightarrow F(x) = \begin{cases} 0, & x\le 0 \\ 1-e^{-\lambda x}, & x>0 \end{cases}$$

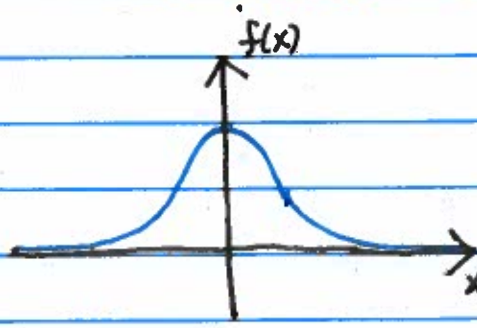


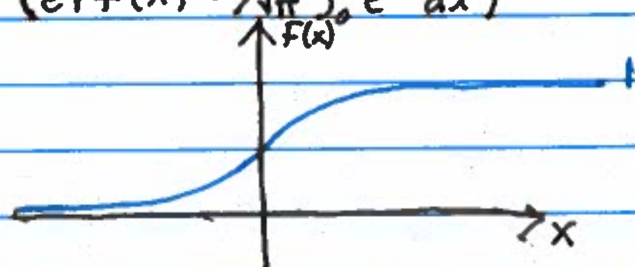


3. Standard Gaussian

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \Rightarrow F(x) = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(x/\sqrt{2}\right)$$

$$\left(\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt\right)$$





Example:

Let X have a standard Gaussian distribution. What is the probability density of X^2?

- If $x<0$, then $P(X^2<x)=0$
- If $x>0$, then $P(X^2<x)$ is given by

$$F_{X^2}(x)=P(X^2<x)$$
$$=P(0<X^2<x)$$
$$=P(0<X<\sqrt{x} \text{ or } -\sqrt{x}<X<0)$$
$$=\frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{x}} \exp\left(\frac{-y^2}{2}\right)dy+\frac{1}{\sqrt{2\pi}}\int_{-\sqrt{x}}^{0} \exp\left(\frac{-y^2}{2}\right)dy$$

Let $u=y^2 \Rightarrow du=2y\,dy=2\sqrt{u}\,du$, $v=y^2 \Rightarrow dv=2y\,dy=-2\sqrt{v}\,dy$

Therefore,

$$F_{X^2}(x)=\frac{1}{\sqrt{2\pi}}\left(\int_0^{x} \exp\left(\frac{-u}{2}\right)\frac{1}{2\sqrt{u}}\,du-\int_x^{0} \exp\left(\frac{-v}{2}\right)\frac{1}{2\sqrt{v}}\,dv\right.$$
$$=\frac{1}{\sqrt{2\pi}}\int_0^{x}\frac{1}{\sqrt{u}}\exp\left(\frac{-u}{2}\right)du$$

Therefore,

$$f_{X^2}(x)=\frac{1}{\sqrt{2\pi x}}\exp\left(\frac{-x}{2}\right).$$