

Lecture #3: Expectation

The expected value of a random variable is simply the weighted average by the probability

Example:

$\Omega = \{R, D, I\}$, $\mathcal{F}$ = set of all subsets

$$X(\omega) = \begin{cases} 1 & \text{if } \omega = R \\ 0 & \text{if } \omega = I \\ -1 & \text{if } \omega = D \end{cases}$$

$$P(X=1) = \frac{1}{2}$$

$$P(X=0) = \frac{1}{6}$$

$$P(X=-1) = \frac{1}{3}$$

$$\begin{aligned} E[X] &= 1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{6} + (-1) \cdot \frac{1}{3} \\ &= \frac{1}{2} - \frac{1}{3} \\ &= \frac{1}{6} \end{aligned}$$

Definition -

$$E[X] = \sum_{\omega \in \Omega} X(\omega) P(\{\omega\})$$

$$= \sum_x x P(X=x)$$

Discrete random
Variable

$$E[X] = \int_{\Omega} X(\omega) dP$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{-\infty}^{\infty} x dF \end{aligned} \quad \left. \begin{array}{l} \text{Continuous} \\ \text{Random} \\ \text{Variable.} \end{array} \right\}$$

Example:

Let X be a random variable with an exponential density,
i.e.,

$$X \sim f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Therefore,

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x \lambda e^{-\lambda x} dx = -x e^{-\lambda x} \Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx = 1/\lambda.$$

Important Expectations

1. Variance: $Var(X) = E[(X - E[X])^2]$

$$\begin{aligned} &= \int_{-\infty}^{\infty} (x - E[X])^2 f(x) dx \\ &= \int_{-\infty}^{\infty} (x^2 - 2x E[X] + E[X]^2) f(x) dx \\ &= \int_{-\infty}^{\infty} x^2 f(x) dx - 2E[X] \int_{-\infty}^{\infty} x f(x) dx + E[X]^2 \int_{-\infty}^{\infty} f(x) dx \\ &= E[X^2] - 2E[X] \cdot E[X] + E[X]^2 \\ &= E[X^2] - E[X]^2 \\ &= \sigma^2 \end{aligned}$$

2. Moments: The n -th moment is $E[X^n] = \int_{-\infty}^{\infty} x^n f(x) dx$.

3. Probability: $E[\mathbb{1}_B] = \int_{-\infty}^{\infty} \mathbb{1}_B(x) f(x) dx$

$$\begin{aligned} &= \int_B f(x) dx \\ &= P(X \in B) \\ &= f_X(B). \end{aligned}$$

4. Moment Generating Function: Let $\lambda \in \mathbb{R}$. The moment generating function is

$$\begin{aligned} \phi(\lambda) &= E[e^{\lambda X}] \\ &= \int_{-\infty}^{\infty} e^{\lambda x} f(x) dx \\ &= \int_{-\infty}^{\infty} (1 + \lambda x + \frac{1}{2} \lambda^2 x^2 + \frac{1}{6} \lambda^3 x^3 + \dots) f(x) dx \end{aligned}$$

$$\Rightarrow \left. \frac{d\phi}{d\lambda} \right|_{\lambda=0} = E[X^n]$$

*Two random variables X, Y have the same distribution if they have the same moment generating function.

Inequalities

1. Markov's Inequality: Let X be a positive random variable on $(\Omega, \mathcal{F}, P)$. Then for any $a > 0$ we have

$$P(X > a) \leq \frac{1}{a} E[X]$$

proof

$$\begin{aligned} E[X] &= \int_0^{\infty} x f(x) dx \\ &= \int_0^a x f(x) dx + \int_a^{\infty} x f(x) dx \\ &\geq \int_a^{\infty} x f(x) dx \\ &\geq \int_a^{\infty} a f(x) dx \\ &= a \int_a^{\infty} f(x) dx \\ &= a P(X > a) \end{aligned}$$

2. Chebyshev's Inequality: For any random variable on $(\Omega, \mathcal{F}, P)$
 $P(|X| > a) \leq \frac{1}{a^2} E[X^2]$

proof

$$P(|X| > a) = P(X^2 > a^2) \leq \frac{1}{a^2} E[X^2]$$

$$\Rightarrow P(|X - E[X]| > n\sigma) \leq \frac{1}{n^2 \sigma^2} E[(X - E[X])^2] = \frac{1}{n^2}$$

$\Rightarrow$ Detecting rare events

n	1	2	3	4	5
$P(X - E[X] > n\sigma)$	< 1	$< 1/4$	$< 1/9 (.11)$	$< 1/16 (.07)$	$< 1/25 (.04)$

3. Chernoff Bound: $P(X > a) \leq e^{-\lambda a} E[e^{\lambda X}]$

proof

$$P(X > a) = P(e^{\lambda X} > e^{\lambda a}) \leq \frac{1}{e^{\lambda a}} E[e^{\lambda X}]$$