

Lecture #4: Random Vectors

On a probability space $(\Omega, \mathcal{F}, P)$ we can have n independent random variables $X_1, \dots, X_n$ such that $X_i: \Omega \rightarrow \mathbb{R}$. This is called a random vector and can also be written $\vec{X}: \Omega \rightarrow \mathbb{R}^n$.

The distribution of $\vec{X}$ is defined by

$$P_{\vec{X}}(A) = P(\vec{X} \in A) \text{ for } A \subseteq \mathbb{R}^n$$

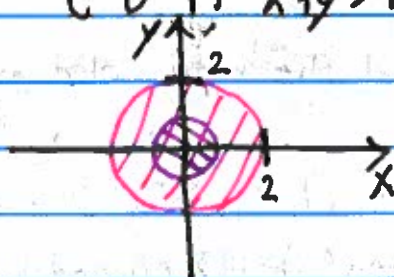
The joint density function (PDF) is defined by

$$P(\vec{X} \in A) = \int_A f(x_1, \dots, x_n) dx_1 \cdots dx_n.$$

Example:

$\vec{X} = (X, Y)$ where $\vec{X}: \Omega \rightarrow \mathbb{R}^2$ is the (x, y) coordinate of a point chosen uniformly in a circle of radius 2.

$$f(x, y) = \begin{cases} \frac{1}{4\pi} & \text{if } x^2 + y^2 \leq 4 \\ 0 & \text{if } x^2 + y^2 > 4 \end{cases}$$

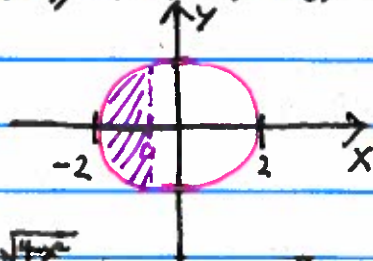


What is $P(\vec{X} \in \{(x, y) \in \mathbb{R}^2: x^2 + y^2 \leq 1\})$?

$$P(\vec{X} \in \{(x, y) \in \mathbb{R}^2: x^2 + y^2 \leq 1\}) = \iint_{x^2 + y^2 \leq 1} \frac{1}{4\pi} dx dy = \frac{1}{4\pi} \int_0^{2\pi} \int_0^1 r dr d\theta = \frac{1}{4}.$$

X, Y are also random variables. What is the PDF of X ?

$$P(X \leq a) = \int_{\{(x,y) \in \mathbb{R}^2 : x \leq a\}} f(x,y) dx dy = F_X(x)$$



$$P(X \leq a) = \int_{-2}^a \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{1}{4\pi} dy dx = \int_{-2}^a \frac{1}{2\pi} \sqrt{4-x^2} dx = F_X(a)$$

$$\Rightarrow \frac{dF_X}{da} = f(a) = \frac{1}{2\pi} \sqrt{4-a^2}$$

The PDF is therefore,

$$f(x) = \begin{cases} \frac{1}{2\pi} \sqrt{4-x^2}, & -2 \leq x \leq 2 \\ 0, & |x| > 2 \end{cases}$$

Similarly, the PDF for Y is given by

$$g(y) = \begin{cases} \frac{1}{2\pi} \sqrt{4-y^2}, & -2 \leq y \leq 2 \\ 0, & |y| > 2 \end{cases}$$

The densities of X, Y are called the marginal densities.

Random variables $X_1, \dots, X_n$ are independent if for all $A_i \in \mathcal{H}$

$$P(X_1 \in A_1, \dots, X_n \in A_n) = P(\bigcap_{i=1}^n X_i \in A_i) = P(X_1 \in A_1 \text{ and } X_2 \in A_2 \text{ and } \dots X_n \in A_n) \\ = P(X_1 \in A_1) P(X_2 \in A_2) \dots P(X_n \in A_n)$$

$$\Rightarrow f(x_1, \dots, x_n) = f(x_1) \dots f(x_n) \leftarrow \text{Product of marginal densities.}$$

Example:

Let (X, Y) be independent, identically distributed (IID) with standard Gaussian distributions.

1. What $P(X^2 + Y^2 < 1)$?

$$P(X^2 + Y^2 < 1) = \iint_{x^2 + y^2 < 1} f(x, y) dx dy$$

$$= \iint_{x^2 + y^2 < 1} \frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2} dx dy$$

$$= \int_0^{2\pi} \int_0^1 \frac{1}{2\pi} e^{-r^2/2} r dr d\theta$$

$$= \int_0^{2\pi} \frac{1}{2\pi} e^{-r^2/2} \Big|_0^1 d\theta$$

$$= \frac{1}{2\pi} (1 - e^{-1/2})$$

2. Let $R = (X^2 + Y^2)^{1/2}$. Then

$$E[R] = \int_0^{2\pi} \int_0^{\infty} \frac{1}{2\pi} e^{-r^2/2} r^2 dr d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} r^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-r^2/2} dr d\theta$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \frac{1}{2} E[X^2] d\theta$$

$$= \frac{2\pi}{\sqrt{2\pi}} \cdot \frac{1}{2}$$

$$= \sqrt{\pi/2}$$

3. What is the PDF of R ?

$$P(R \leq r) = \int_0^{2\pi} \int_0^r \frac{1}{2\pi} e^{-s^2/2} s ds d\theta$$

$$= \int_0^{2\pi} e^{-s^2/2} s ds$$

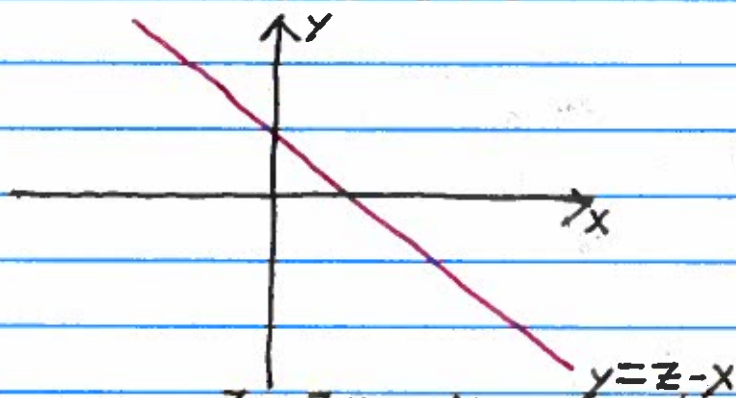
$$\Rightarrow f(r) = re^{-r^2/2}$$

Example:

Let X, Y be two independent Gaussian random variables with mean 0 and standard deviation σ_X, σ_Y . What is the PDF of $X+Y=Z$?

One Approach:

$$P(Z_1 \leq z) = P(X+Y \leq z) \\ = P(Y \leq z-X)$$



$$P(Z_1 \leq z) = \int_{-\infty}^z \int_{-\infty}^{z-x} \frac{e^{-x^2/2\sigma_X^2 - y^2/2\sigma_Y^2}}{2\pi\sigma_X\sigma_Y} dy dx \quad (\text{Not that helpful!})$$

Another Approach:

What if Z is Gaussian? Then, we just need to know the following quantities:

$$\mu = E[Z] \quad \text{and} \quad \sigma^2 = E[(Z - E[Z])^2] = E[Z^2] - E[Z]^2$$

First,

$$E[Z] = E[X+Y] = E[X] + E[Y] = 0.$$

Second,

$$E[Z^2] = E[(X^2 + 2XY + Y^2)] \\ = E[X^2] + 2E[XY] + E[Y^2] \\ = \sigma_X^2 + 2E[X] \cdot E[Y] + \sigma_Y^2 \\ = \sigma_X^2 + \sigma_Y^2.$$

If all of this is true then

$$Z \sim \frac{1}{\sqrt{2\pi(\sigma_x^2 + \sigma_y^2)}} \exp\left(-\frac{z^2}{2(\sigma_x^2 + \sigma_y^2)}\right)$$

* This can be shown using moment generating functions but we first need some facts *

Fact #1: If X, Y are independent random variables then $E[X \cdot Y] = E[X] \cdot E[Y]$.

proof:

$$\begin{aligned} E[X \cdot Y] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot y f(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot y g(x) h(y) dx dy \\ &= \int_{-\infty}^{\infty} x g(x) \left(\int_{-\infty}^{\infty} y h(y) dy \right) dx \\ &= \int_{-\infty}^{\infty} x g(x) E[Y] dx \\ &= E[Y] \int_{-\infty}^{\infty} x g(x) dx \\ &= E[Y] \cdot E[X]. \end{aligned}$$

Fact #2: If X, Y are independent and $f: \mathbb{R} \rightarrow \mathbb{R}$ is measurable then $f(X), f(Y)$ are independent.

proof:

$$\begin{aligned} P(f(X) \in A, f(Y) \in B) &= P(\{\omega: X(\omega) \in f^{-1}(A), Y(\omega) \in f^{-1}(B)\}) \\ &= P(\{\omega: X(\omega) \in f^{-1}(A)\}) P(\{\omega: Y(\omega) \in f^{-1}(B)\}) \\ &= P(f(X) \in A) \cdot P(f(Y) \in B) \end{aligned}$$

Fact #3:

If X is Gaussian with mean μ and standard deviation σ then

$$\phi(\lambda) = \mathbb{E}[e^{\lambda X}] = e^{\lambda\mu + \lambda^2\sigma^2/2}$$

proof:

Done in homework.

^{independent}

Theorem: If X, Y are Gaussian random variables with mean μ_X, μ_Y and standard deviations σ_X, σ_Y then $X+Y$ is Gaussian.

proof (Sketch):

$$\phi(\lambda) = \mathbb{E}[\exp(\lambda(X+Y))] = \mathbb{E}[\exp(\lambda X)\exp(\lambda Y)] = \mathbb{E}[e^{\lambda X}] \mathbb{E}[e^{\lambda Y}]$$

$$\Rightarrow \phi(\lambda) = e^{\lambda\mu_X + \lambda^2\sigma_X^2/2} e^{\lambda\mu_Y + \lambda^2\sigma_Y^2/2}$$

$$= \exp(\lambda(\mu_X + \mu_Y) + \lambda^2(\sigma_X^2 + \sigma_Y^2)/2)$$

Therefore, $X+Y$ is Gaussian with $\mu = \mu_X + \mu_Y$, $\sigma = \sqrt{\sigma_X^2 + \sigma_Y^2}$.