

Lecture #6: Gaussian Processes

Stochastic Process

$$X = (X_t, t \in \mathcal{T}), \text{ on } (\Omega, \mathcal{F}, P)$$

where $\mathcal{T}$ is discrete or continuous, i.e.,

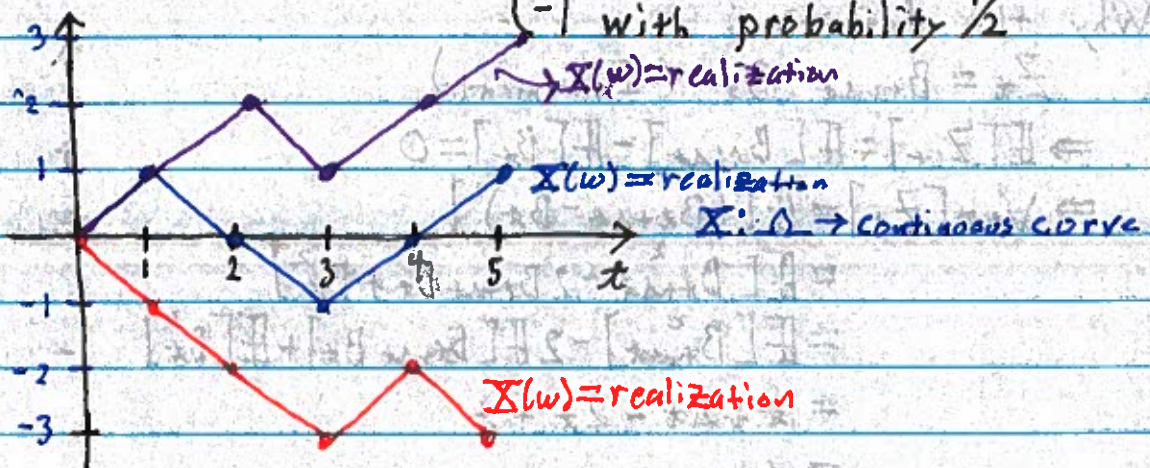
$$\mathcal{T} = \{0, 1, 2, 3, \dots\} \text{ or } \mathcal{T} = [0, \infty).$$

Examples:

1. $\mathcal{T} = \{0, 1, 2, 3, \dots\}$

$$X_0 = 1$$

(Increment) $X_{t+1} - X_t = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases}$



2. $\mathcal{T} = [0, \infty)$

$$X_0 = 0$$

$$X_{t+\Delta t} - X_t \sim N(0, \sigma^2 \Delta t) \quad (\text{Gaussian Increment})$$

$$\Rightarrow X_{t+\Delta t} = X_t + \sigma \sqrt{\Delta t} Z, \quad Z \leftarrow \text{standard Gaussian}$$



Definition - A Gaussian process $X = (X_t, t \in \mathcal{T})$ is a stochastic process which satisfy for all $n \in \mathbb{N}$ and any $t_1, \dots, t_n$ the random vector $(X_{t_1}, \dots, X_{t_n})$ is Gaussian. The distribution of X is determined by

$$m(t) = \mathbb{E}[X_t] \quad \leftarrow \text{Defined by a physical process}$$

$$C(s, t) = \text{Cov}(X_s, X_t) \quad \leftarrow \text{physical process}$$

Examples:

1. Brownian Motion:

$(B_t, t \geq 0)$ is the Gaussian process with

$$\bullet m(t) = \mathbb{E}[B_t] = 0$$

$$\bullet C(s, t) = \text{Cov}(B_s, B_t) = \min\{s, t\}$$

Why this definition?

$$Z_t = B_{t+\Delta t} - B_t \quad (\text{Increment})$$

$$\Rightarrow \mathbb{E}[Z_{t+\Delta t}] = \mathbb{E}[B_{t+\Delta t}] - \mathbb{E}[B_t] = 0$$

$$\Rightarrow \text{Var}[Z_t] = \mathbb{E}[(B_{t+\Delta t} - B_t)^2]$$

$$= \mathbb{E}[B_{t+\Delta t}^2 - 2B_{t+\Delta t}B_t + B_t^2]$$

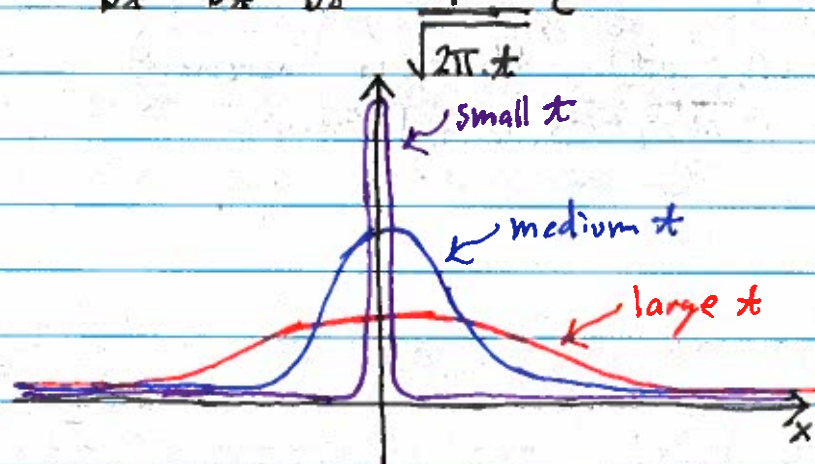
$$= \mathbb{E}[B_{t+\Delta t}^2] - 2\mathbb{E}[B_{t+\Delta t}B_t] + \mathbb{E}[B_t^2]$$

$$= t + \Delta t - 2t + t$$

$$= \Delta t.$$

If we know B_0 , i.e. $B_0 = 0$ then

$$B_t = B_t - B_0 \sim \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$



We can also generate realizations: $B_{t+\Delta t} = B_t + \sqrt{\Delta t} N(0, 1)$.

2. Brownian Motion with Drift

$$\text{Let } X_t = \sigma B_t + \nu t$$

$$\Rightarrow \mathbb{E}[X_t] = \nu t$$

$$\Rightarrow \text{Cov}(X_t, X_s) = \text{Cov}(\sigma B_t + \nu t, \sigma B_s + \nu s)$$

$$\begin{aligned} &= \text{Cov}(\sigma B_t, \sigma B_s) + \text{Cov}(\sigma B_t, \nu s) + \text{Cov}(\nu t, \sigma B_s) \\ &\quad + \text{Cov}(\nu t, \nu s) \\ &= \sigma^2 \text{Cov}(B_t, B_s) + \sigma \nu \text{Cov}(B_t, s) + \nu \sigma \text{Cov}(t, B_s) \\ &\quad + \nu^2 \text{Cov}(t, s) \end{aligned}$$

Note:

$$\bullet \text{Cov}(B_t, s) = \mathbb{E}[s B_t] - \mathbb{E}[s] \mathbb{E}[B_t] = s \mathbb{E}[B_t] - s \mathbb{E}[B_t] = 0$$

$$\bullet \text{Cov}(t, B_s) = 0$$

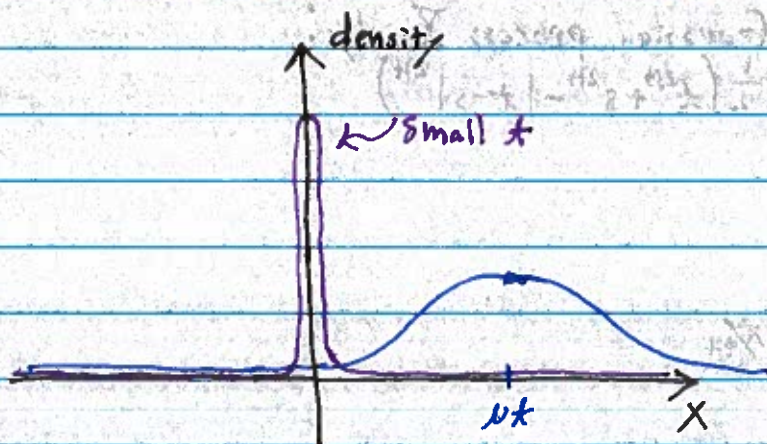
$$\bullet \text{Cov}(t, s) = \mathbb{E}[ts] - \mathbb{E}[s] \mathbb{E}[t] = ts - st = 0$$

Therefore,

$$\text{Cov}(X_t, X_s) = \sigma^2 \min\{s, t\}$$

Consequently,

$$X_t \sim \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x - \nu t)^2}{2\sigma^2}}$$



We also have

$$X_{t+\Delta t} - X_t = \sigma B_{t+\Delta t} + \nu(t+\Delta t) - \sigma B_t - \nu t$$

$$\Rightarrow X_{t+\Delta t} = X_t + \sigma(B_{t+\Delta t} - B_t) + \nu \Delta t$$

$$\Rightarrow X_{t+\Delta t} = X_t + \sigma N(0, \Delta t) + \nu \Delta t$$

$$\Rightarrow X_{t+\Delta t} = X_t + \sigma \sqrt{\Delta t} N(0, 1) + \nu \Delta t$$

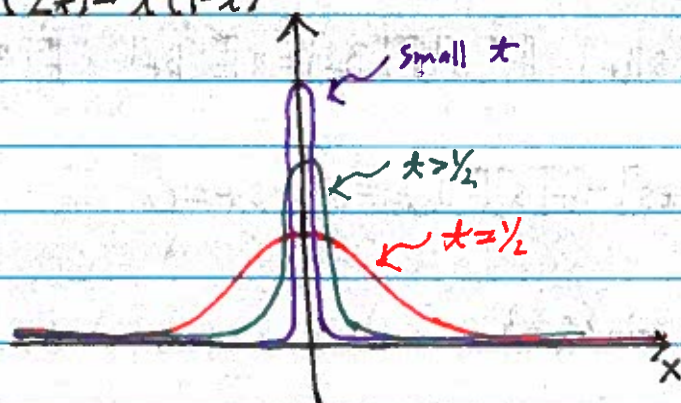
3. Brownian Bridge

$$Z_t = B_t - t B_1, \quad t \in [0, 1]$$

$$\Rightarrow \mathbb{E}[Z_t] = 0$$

$$\begin{aligned} \text{Cov}(Z_t, Z_s) &= \text{Cov}(B_t - t B_1, B_s - s B_1) \\ &= \text{Cov}(B_t, B_s) - t \text{Cov}(B_t, B_1) + t s \text{Cov}(B_1, B_1) - s \text{Cov}(B_t, B_1) \\ &= \min\{t, s\} - s t \\ &= \begin{cases} t(1-s), & \text{if } t < s \\ s(1-t), & \text{if } t \geq s \end{cases} \end{aligned}$$

$$\text{Var}(Z_t) = t(1-t)$$



We also have!

$$Z_{t+\Delta t} - Z_t = B_{t+\Delta t} - B_t - \Delta t B_1$$

$$\Rightarrow Z_{t+\Delta t} = Z_t - \Delta t B_1 + \sqrt{\Delta t} B(0, 1)$$

4. Fractional Brownian Motion

Let $0 < H < 1$. Define Gaussian process Y_t by:

$$\bullet \text{Cov}(Y_s, Y_t) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$$

$$\bullet \mathbb{E}[Y_t] = 0$$

Therefore,

$$\text{Var}(Y_t) = t^{2H}$$

$$\Rightarrow Y_t \sim \frac{1}{\sqrt{2\pi} t^H} e^{-\frac{x^2}{2t^{2H}}}$$

Let $Z_t = Y_{t+\Delta t} - Y_t$. Therefore,

$$\begin{aligned} \text{Var}(Z_t) &= \text{Cov}(Y_{t+\Delta t} - Y_t, Y_{t+\Delta t} - Y_t) \\ &= \text{Cov}(Y_{t+\Delta t}, Y_{t+\Delta t}) - 2\text{Cov}(Y_{t+\Delta t}, Y_t) + \text{Cov}(Y_t, Y_t) \\ &= (t+\Delta t)^{2H} - t^{2H} - (t+\Delta t)^{2H} + \Delta t^{2H} + t^{2H} \\ &= \Delta t^{2H} \end{aligned}$$

$$\Rightarrow Y_{t+\Delta t} = Y_t + \Delta t^H N(0, 1)$$