WFU Analysis Seminar: Nonlinear Analysis

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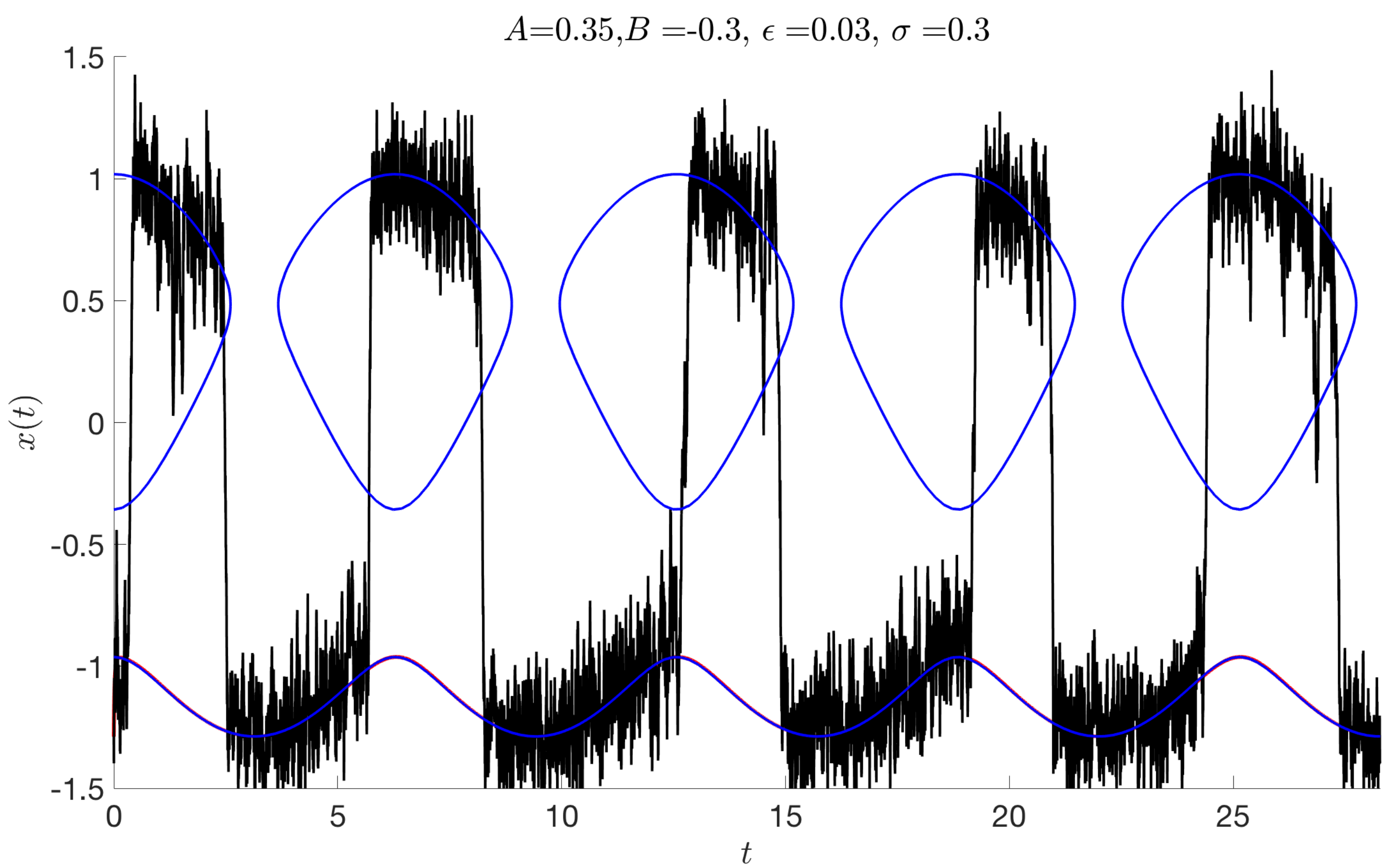
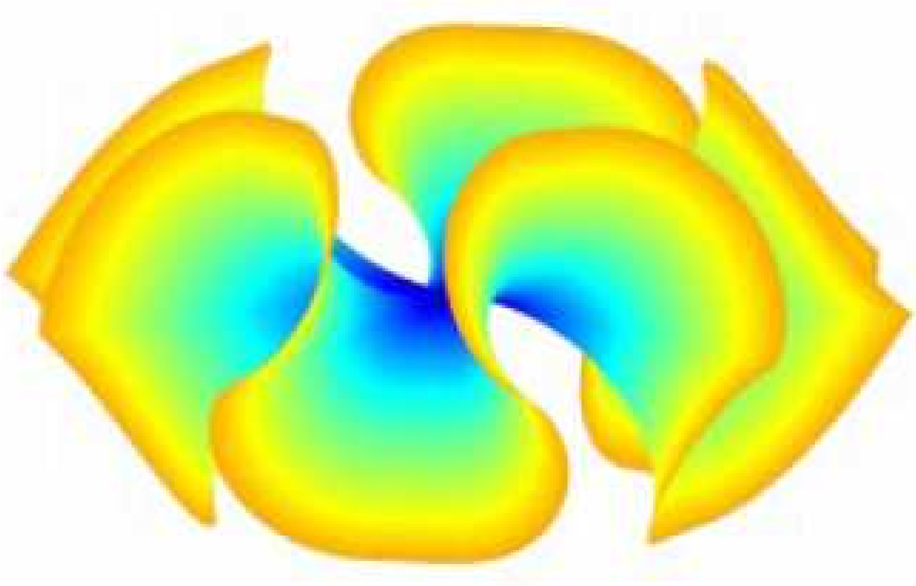


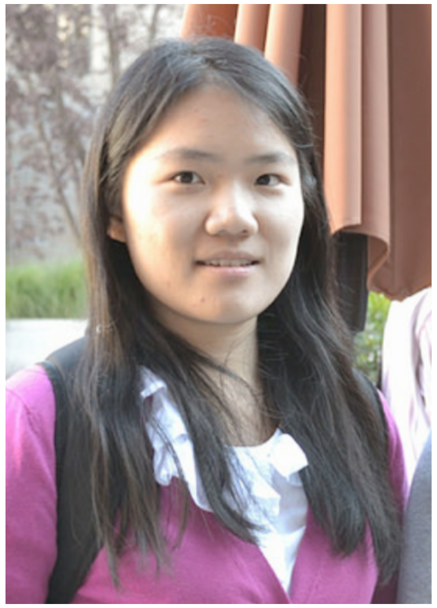


WAKE FOREST UNIVERSITY

Collaborators


Max Rezek


Jessica Zanetell


Yuxin Chen


Alexandria Volkening


Shankar


Bjorn Sandstede


Mary Silber

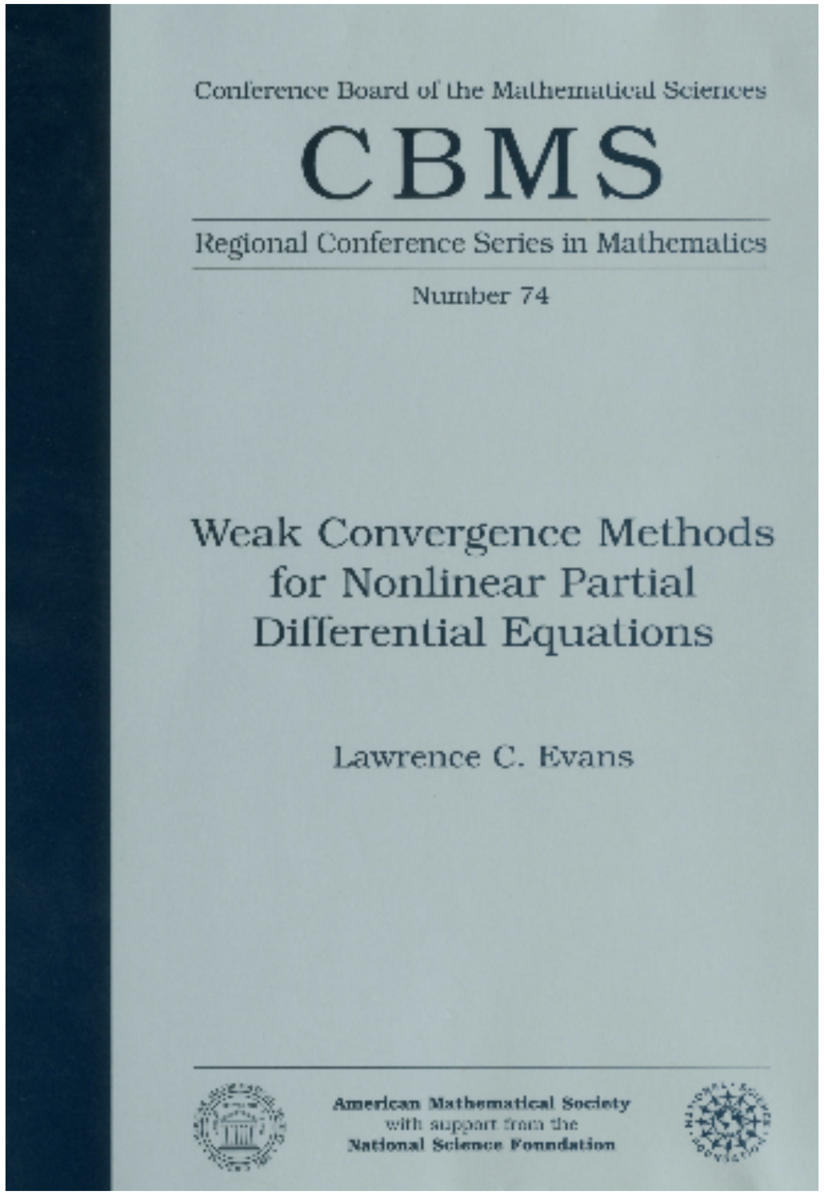
Outline

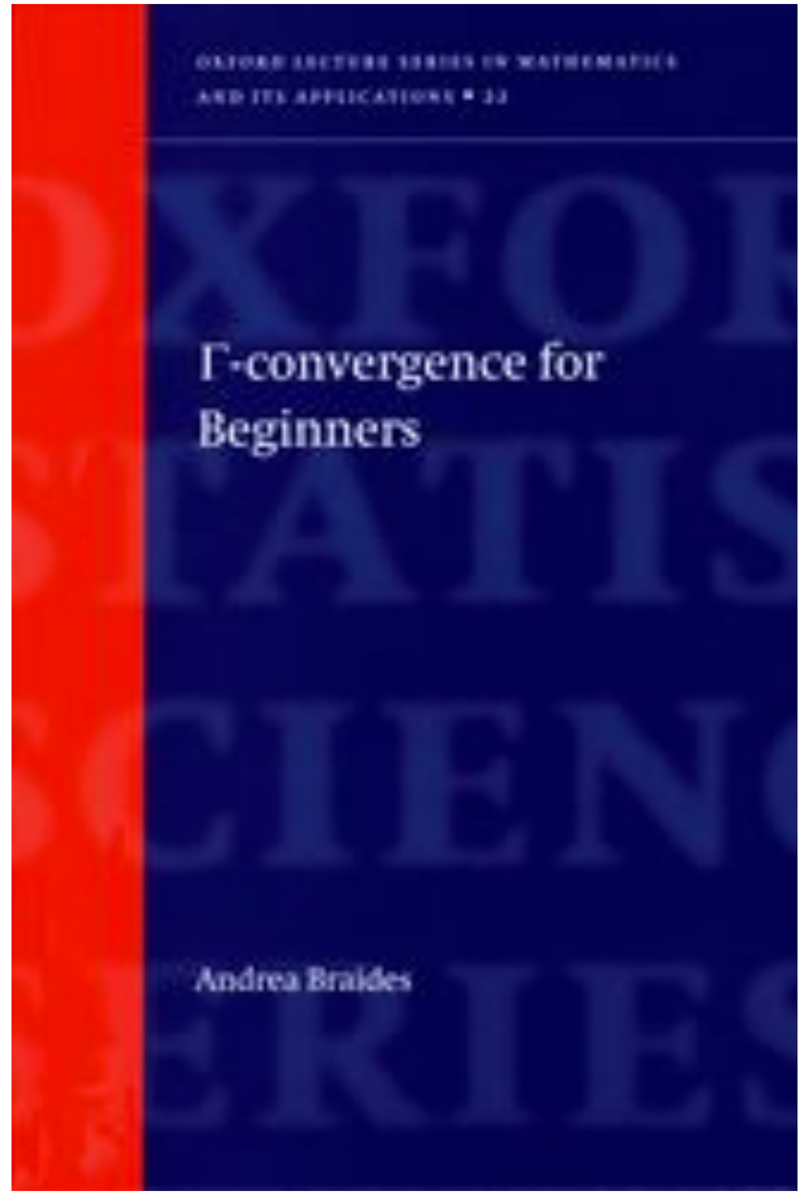
1. Lecture 1 (2/22/19) **Survey of Nonlinear Analysis:**
   a. Intro to nonlinear methods
   b. Intro to some modern problems in calculus of variations
   c. Intro to Gamma convergence

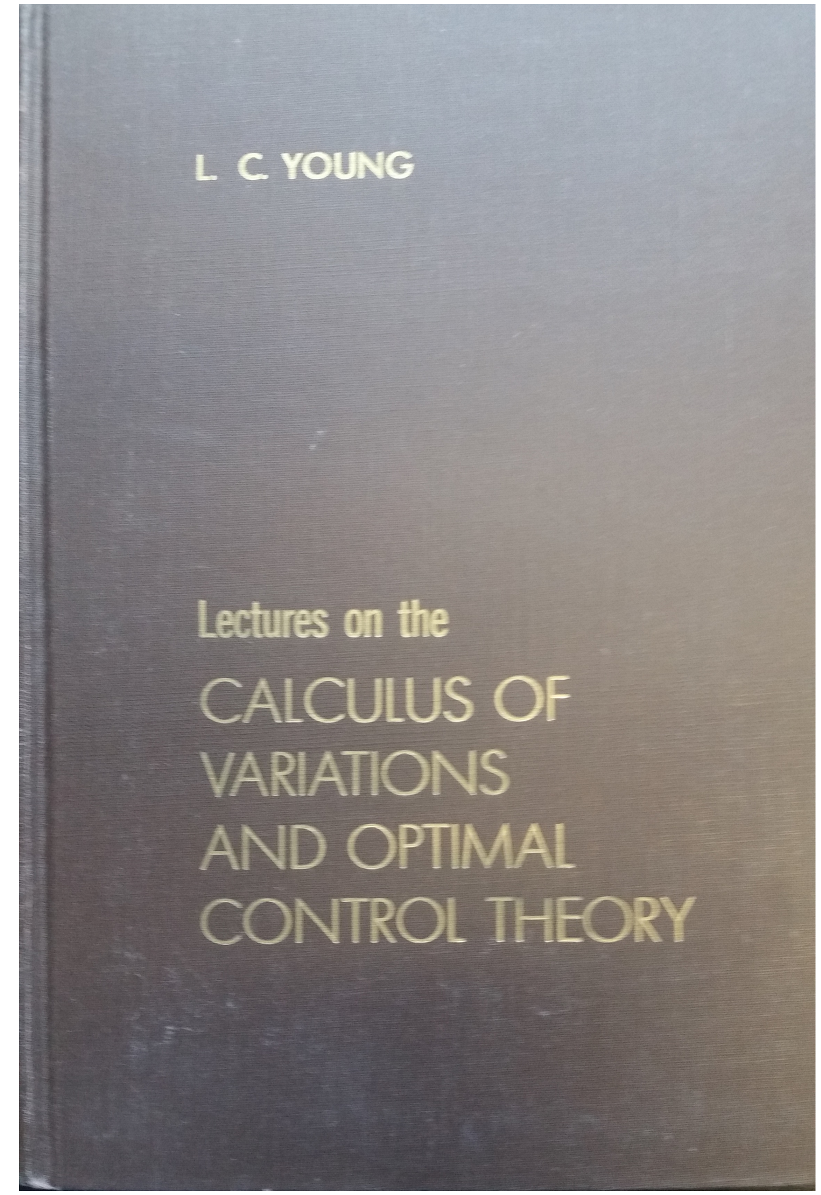
2. Lecture 2 (3/01/19) **Pattern Formation in Hyperbolic Sheets:**
   a. Intro to non-Euclidean model of thin elastic sheets
   b. Existence and regularity of (some) solutions to Monge-Ampere equation

3. Lecture 3 (3/08/19) **Large Deviations in SDES:**
   a. Intro to Friedlin-Wentzell functional
   b. Tipping events in piecewise-smooth SDEs.

Books!





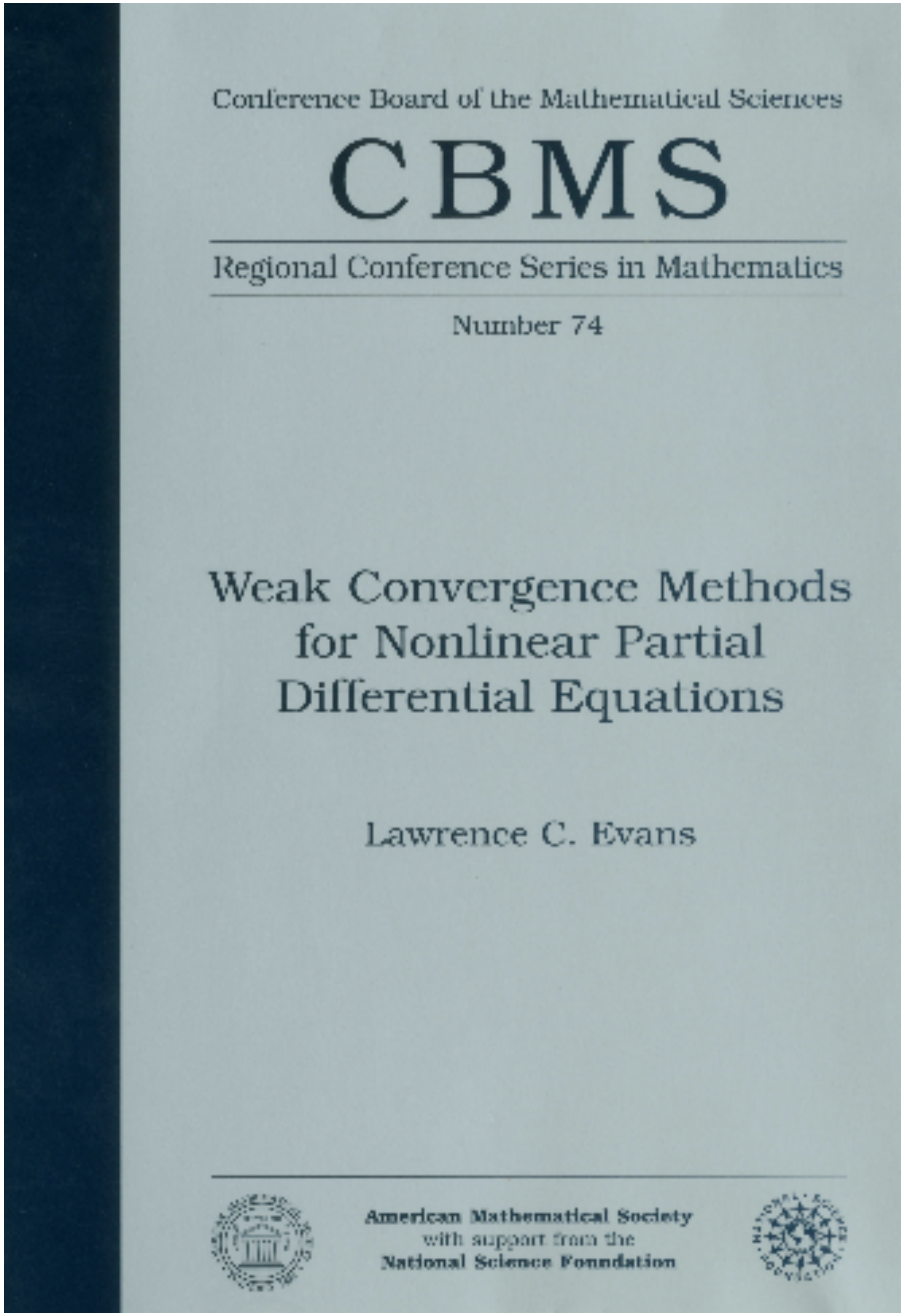


Evans, L. C. (1990). *Weak convergence methods for nonlinear partial differential equations* (No. 74). American Mathematical Soc..

Braides, A. (2002). *Gamma-convergence for Beginners* (Vol. 22). Clarendon Press.

Young, L. C. (2000). Lecture on the calculus of variations and optimal control theory (Vol. 304). American Mathematical Soc..

Big Picture



Solve the following equation:

$$N[u] = f$$

Idea: Solve approximate problem

$$N_k[u_k] = f_k$$

Goal: Prove the following

$$u_k \to u^*$$

$$N[u^*] = f$$

Problem: Weak Compactness

$$u_k \rightharpoonup u^*$$

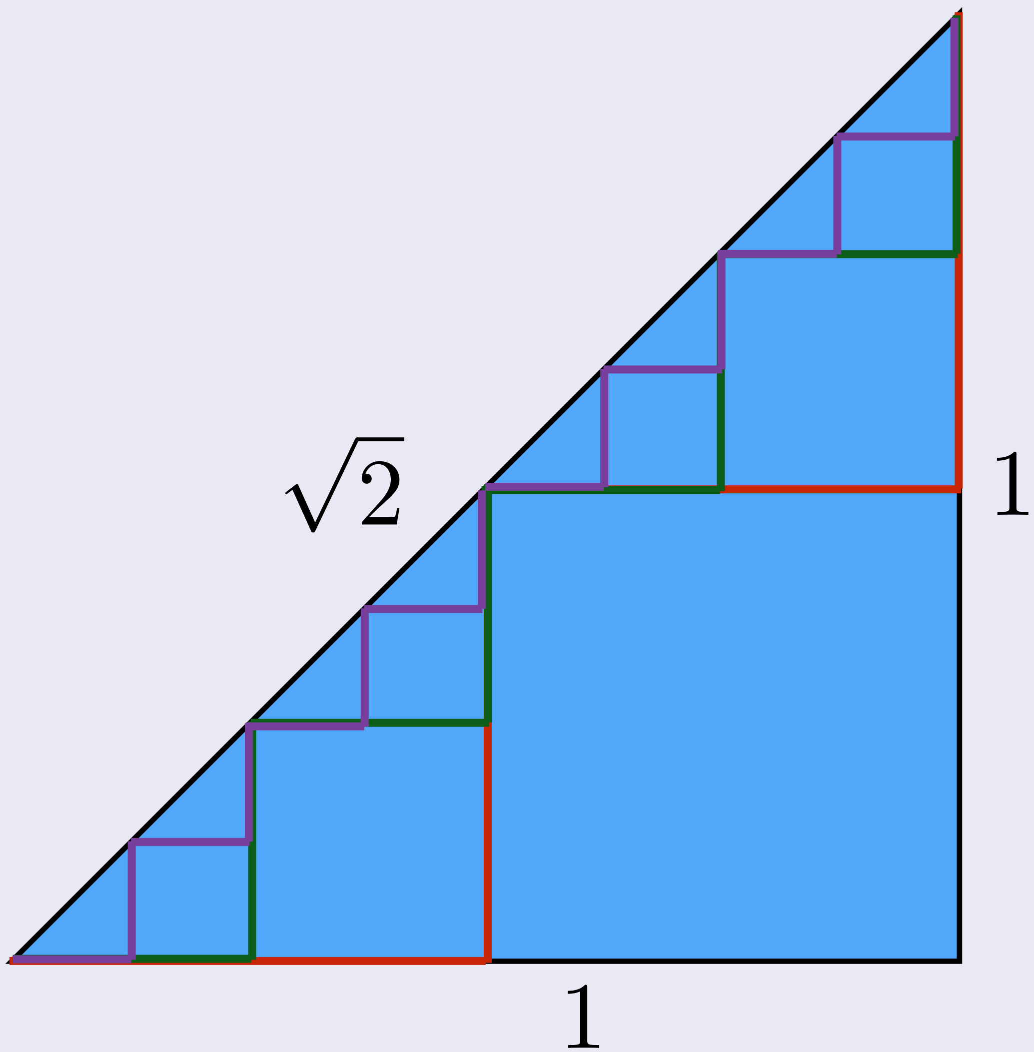
$$N[u_k] \not\rightharpoonup N[u^*]$$

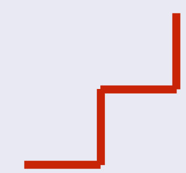
Why Care?

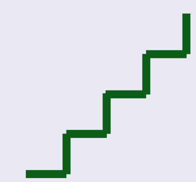
Theorem

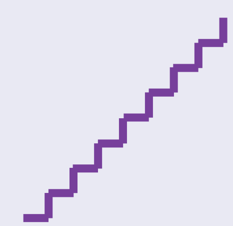
$$\sqrt{2} = 2$$

Proof.

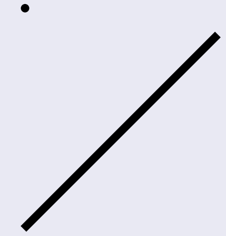


$f_1 =$  $L[f_1] = 2$

$f_2 =$  $L[f_2] = 2$

$f_3 =$  $L[f_3] = 2$

$\vdots$ $\vdots$

$f =$  $L[f] = 2$

□

The World's First Nonlinear Problem

Your "mathematical reality" is $\mathbb{Q}$, solve the following equation:

$$x^2 = 2$$

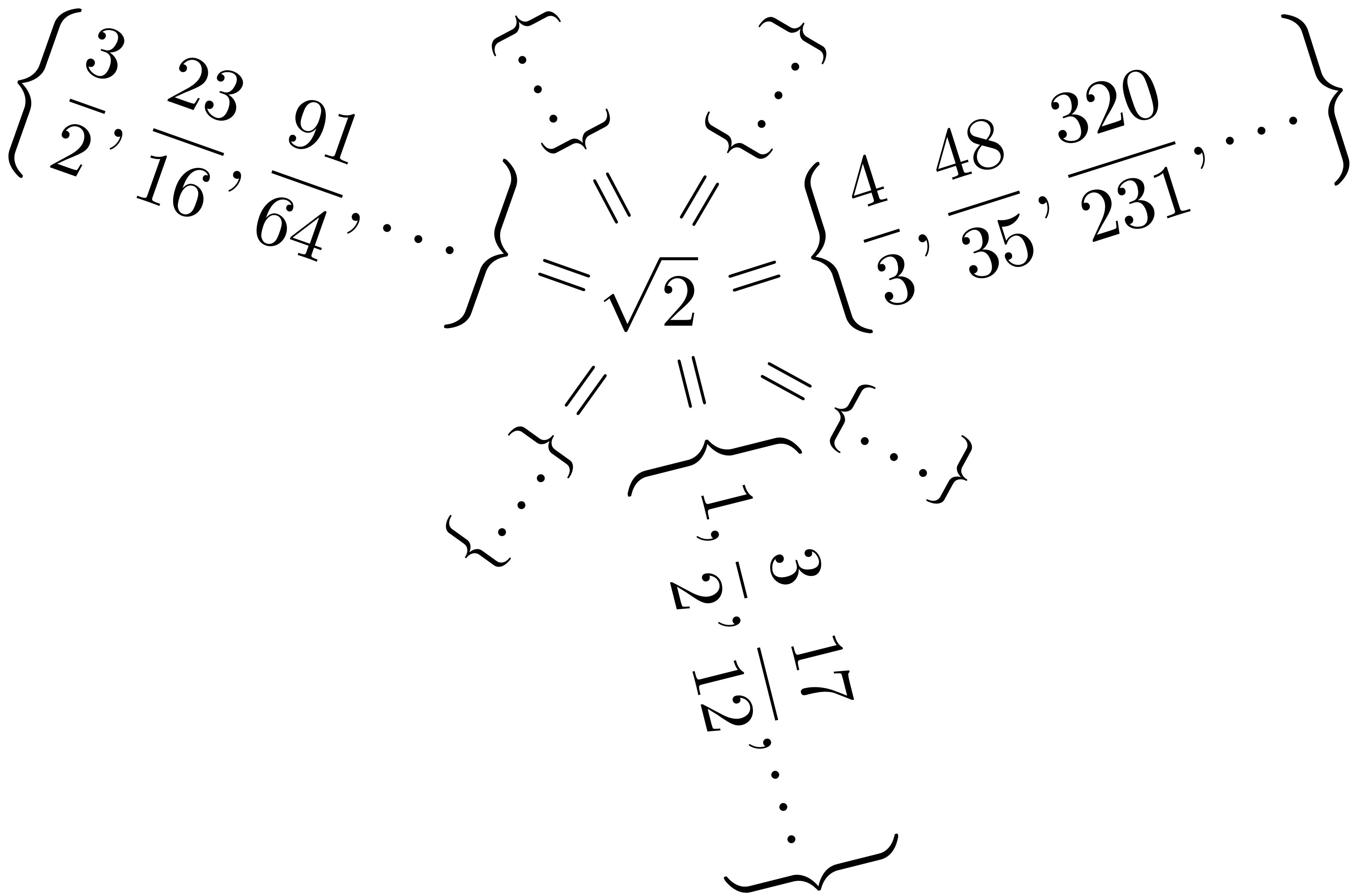
Approximation Schemes:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \qquad \{x_n\}_{n=1}^{\infty} = \left\{1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \ldots\right\}$$

$$x_n = \sum_{m=1}^{n} \frac{(2m)!}{(1-2m)(m!)^2 2^{3m-1}} \qquad \{x_n\}_{n=1}^{\infty} = \left\{\frac{3}{2}, \frac{23}{16}, \frac{91}{64}, \ldots\right\}$$

$$x_n = \prod_{m=0}^{n} \frac{(4m+2)^2}{(4m+1)(4m+3)} \qquad \{x_n\}_{n=1}^{\infty} = \left\{\frac{4}{3}, \frac{48}{35}, \frac{320}{231}, \ldots\right\}$$

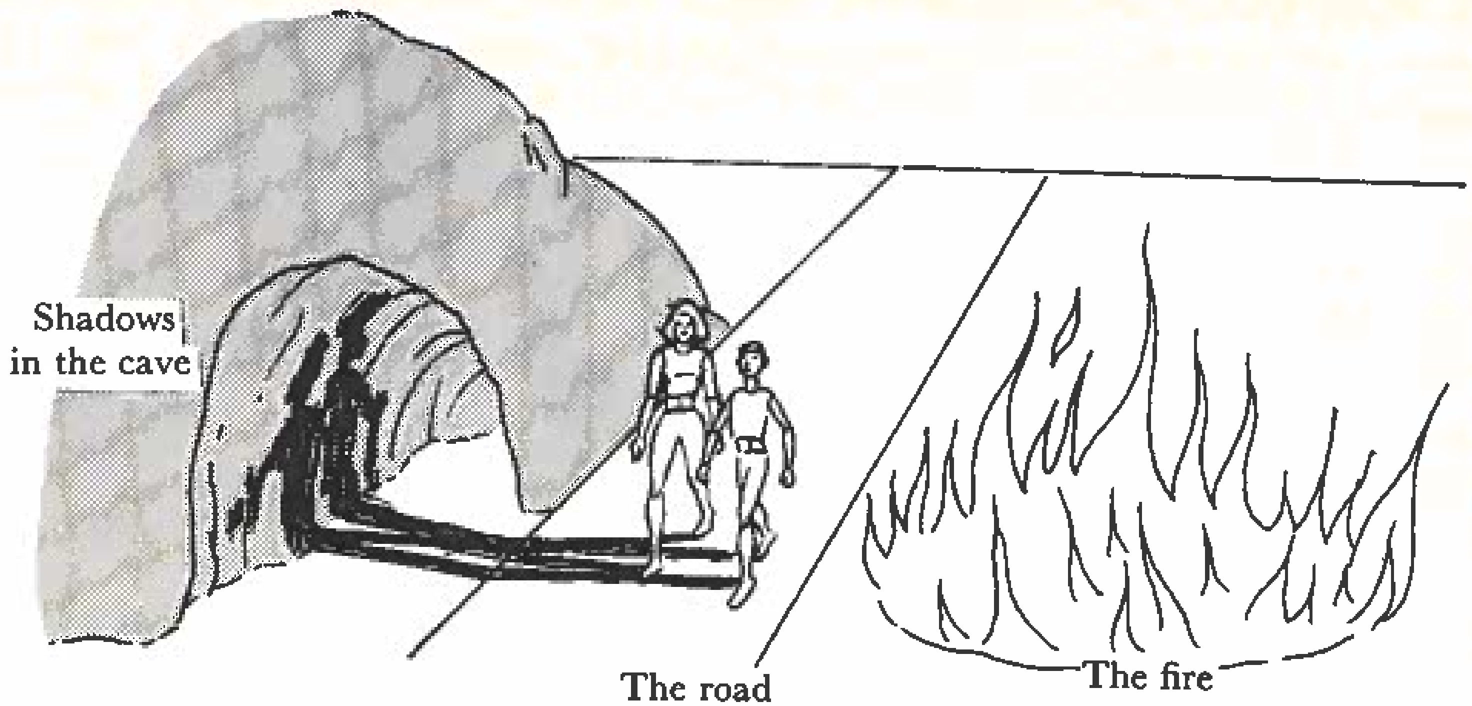
Solution to Square Root Problem



$\sqrt{2}$ is an equivalence class of approximations schemes! That is, it is an **equivalence class of Cauchy sequences.**

Equivalence relation: $\lim |x_n - y_n| = 0.$

We All Live in a Cave



The classical concept of a curve C corresponds very closely to the curves that we see and draw; and such curves can twist and turn and zigzag back and forth; moreover they can have highly complicated self-intersections. This is only the shadow. If we choose to substitute for it the notion of curvilinear integral $\mathscr{I}(C)$, it will become an element g of our dual space $\mathscr{C}_0^*(A)$. This element, of course, we cannot see directly, but only by its shadow C. However, it is g and not C which obeys our criterion of simplicity, for g is a linear function of the new variable $f \in \mathscr{C}_0(A)$.

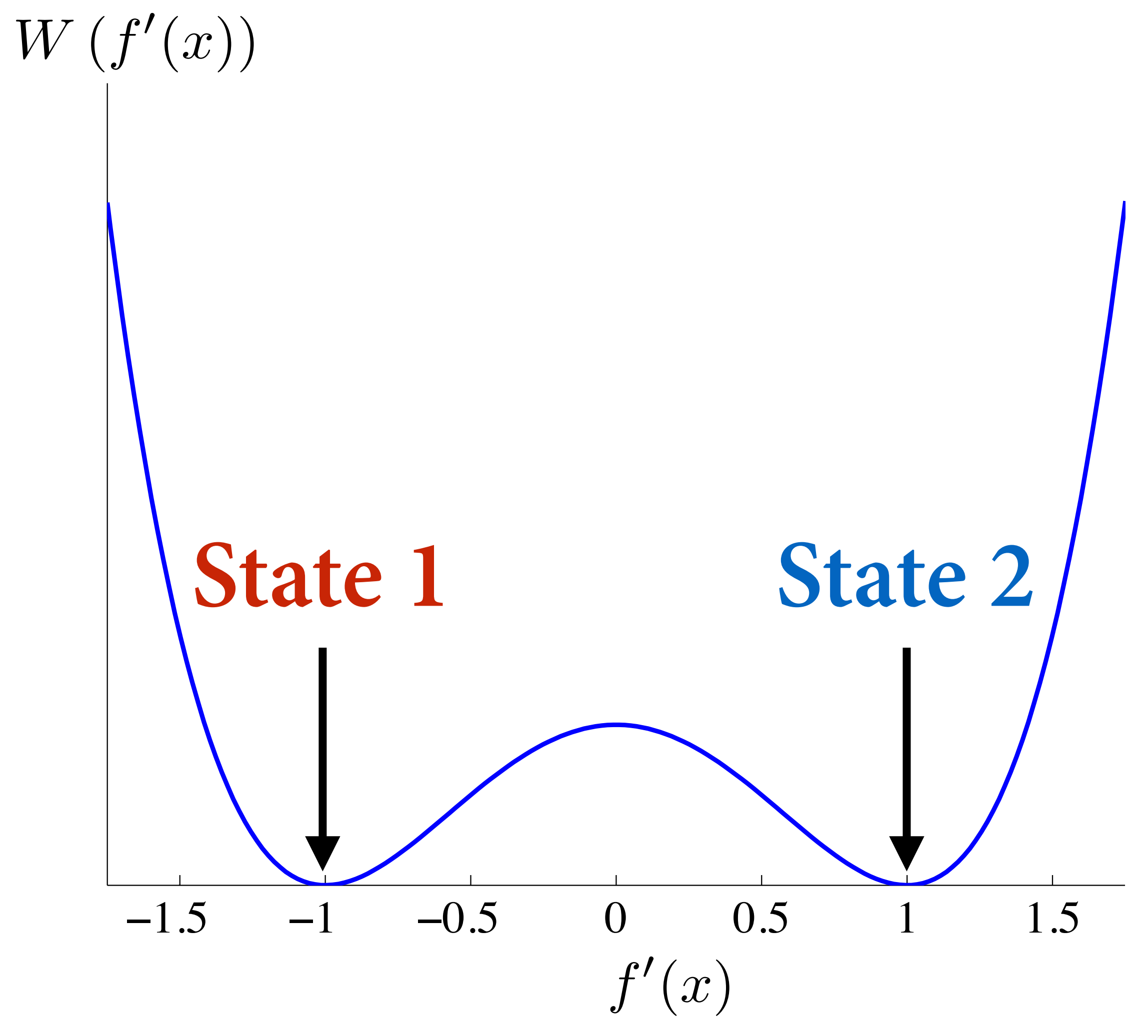
§65. A HUMAN ANALOGY

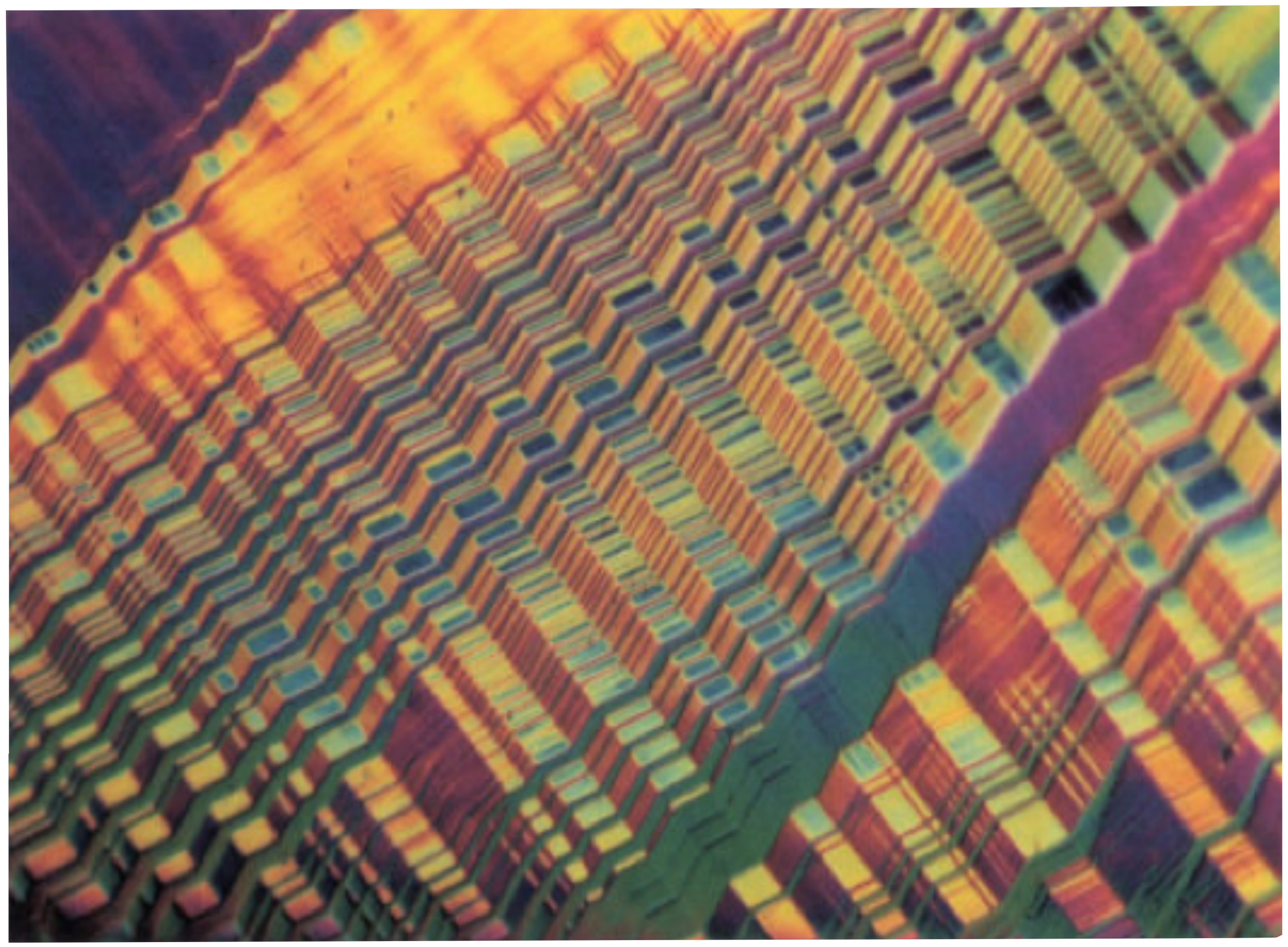
Calculus of Variations Example 1

$$E[f] = \int_0^1 W\left(f'(x)\right)\,dx = \int_0^1 \left(\left(\frac{df}{dx}\right)^2 - 1\right)^2 dx$$

$$f(0) = f(1) = 0$$

Goal: Find f that minimizes the *functional* E.





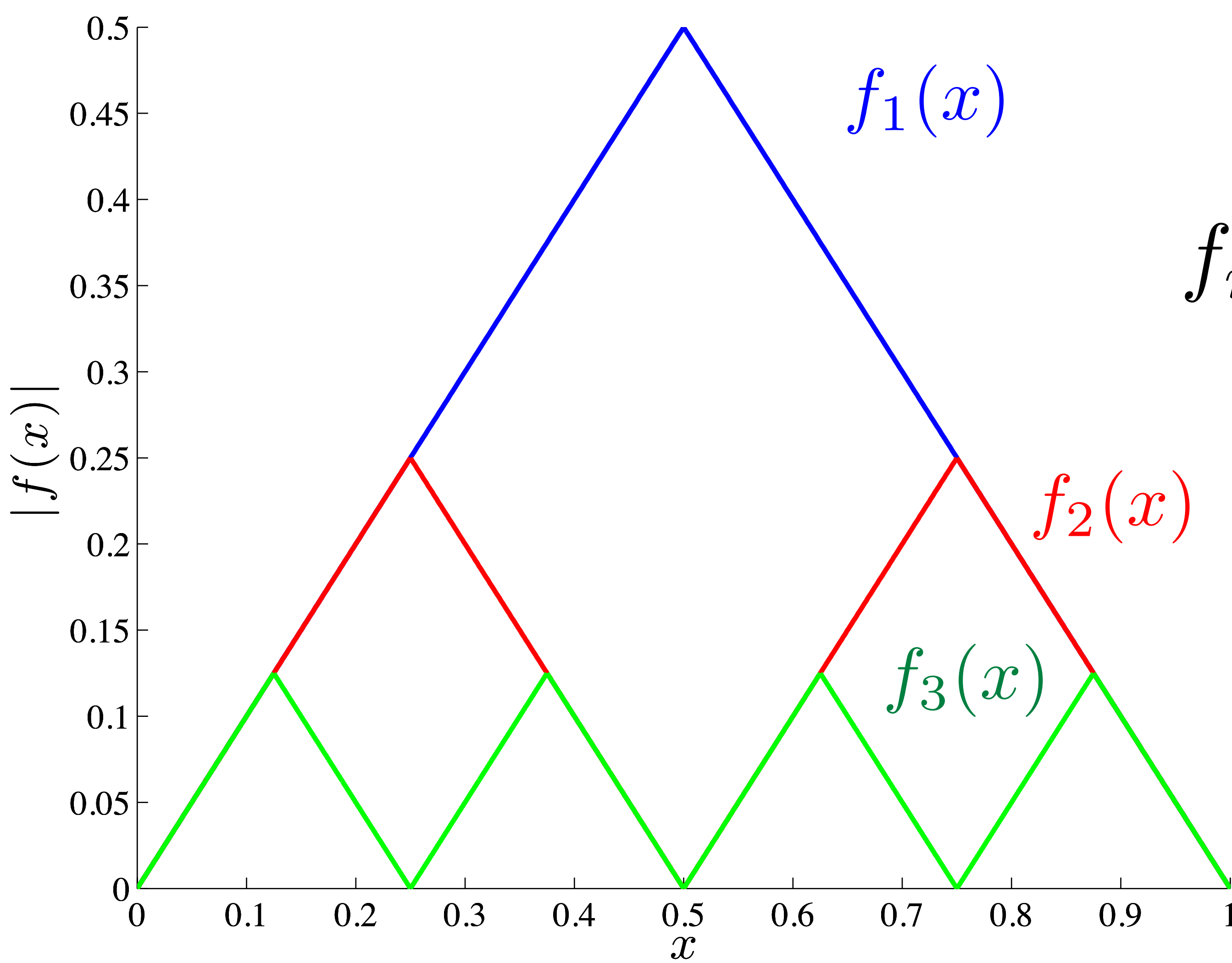
Infinite number of minimizers!

Calculus of Variations Example 2

$$E[f] = \int_0^1 f(x)^2\,dx + \int_0^1 \left(\left(\frac{df}{dx}\right)^2 - 1\right)^2 dx$$

Vanishes when $f = 0$

Vanishes when $f' = \pm 1$



$f_n \to 0$ and $E[f_n] \to 0 \neq E[0]$.

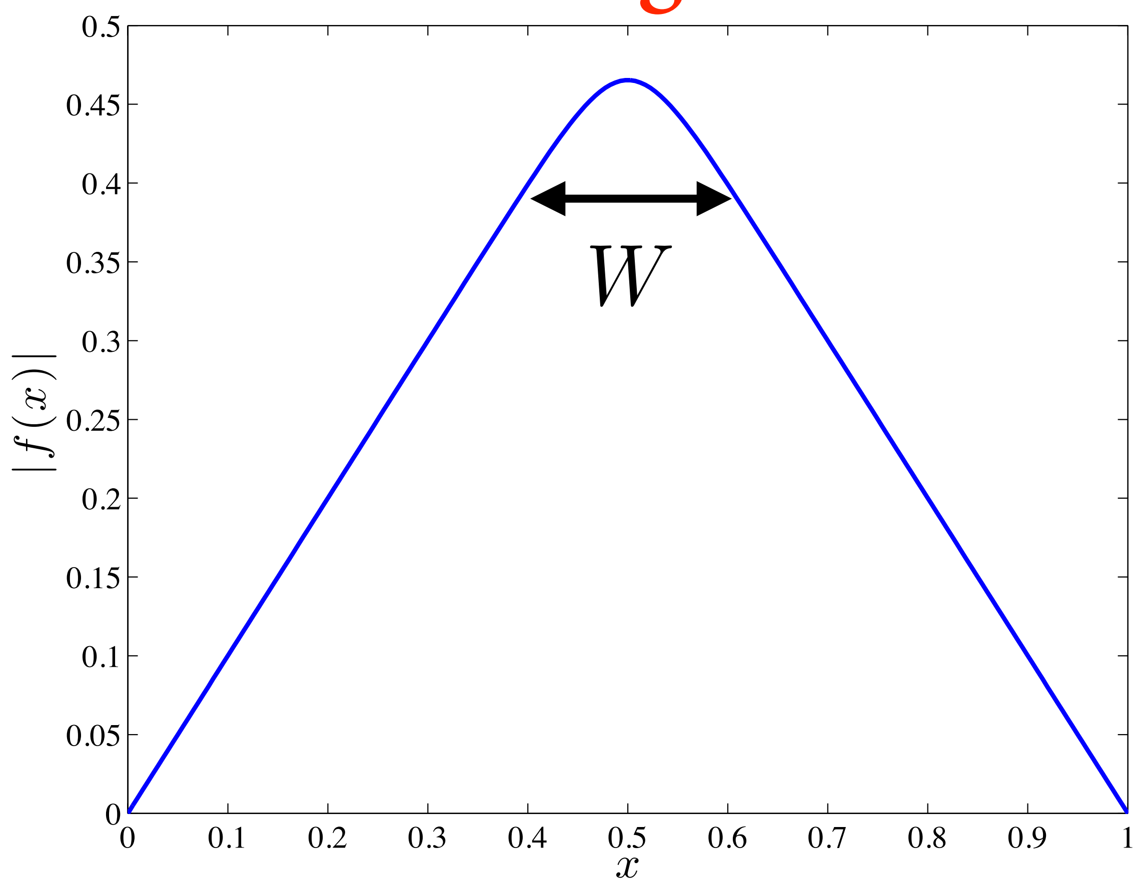
There is no minimum for this problem!

Minimizing Sequence

Calculus of Variations Example 3

$$E_\epsilon[f] = \frac{1}{\epsilon}\int_0^1 \left(\left(\frac{df}{dx}\right)^2 - 1\right)^2 dx + \epsilon \int_0^1 \left(\frac{d^2 f}{dx^2}\right)^2 dx.$$

Good guess:



$$E_\epsilon[f_W] \approx \frac{1}{\epsilon}\int_0^W \frac{1}{4}\,dx + \epsilon \int_0^W \frac{4}{W^2}\,ds$$

$$= \frac{W}{4\epsilon} + \frac{4\epsilon}{W}$$

Optimize over W:

$$W = \epsilon$$

$$\Rightarrow \min E_\epsilon[f] \leq C.$$

Limiting Energy

$$\lim_{\epsilon \to 0} E_\epsilon[f] = \begin{cases} \frac{8}{3}\#\text{ of triangles}, & \text{if } f = \pm 1 \text{ a.e.} \\ \infty, & \text{o.w.} \end{cases}$$

Calculus of Variations Example 3

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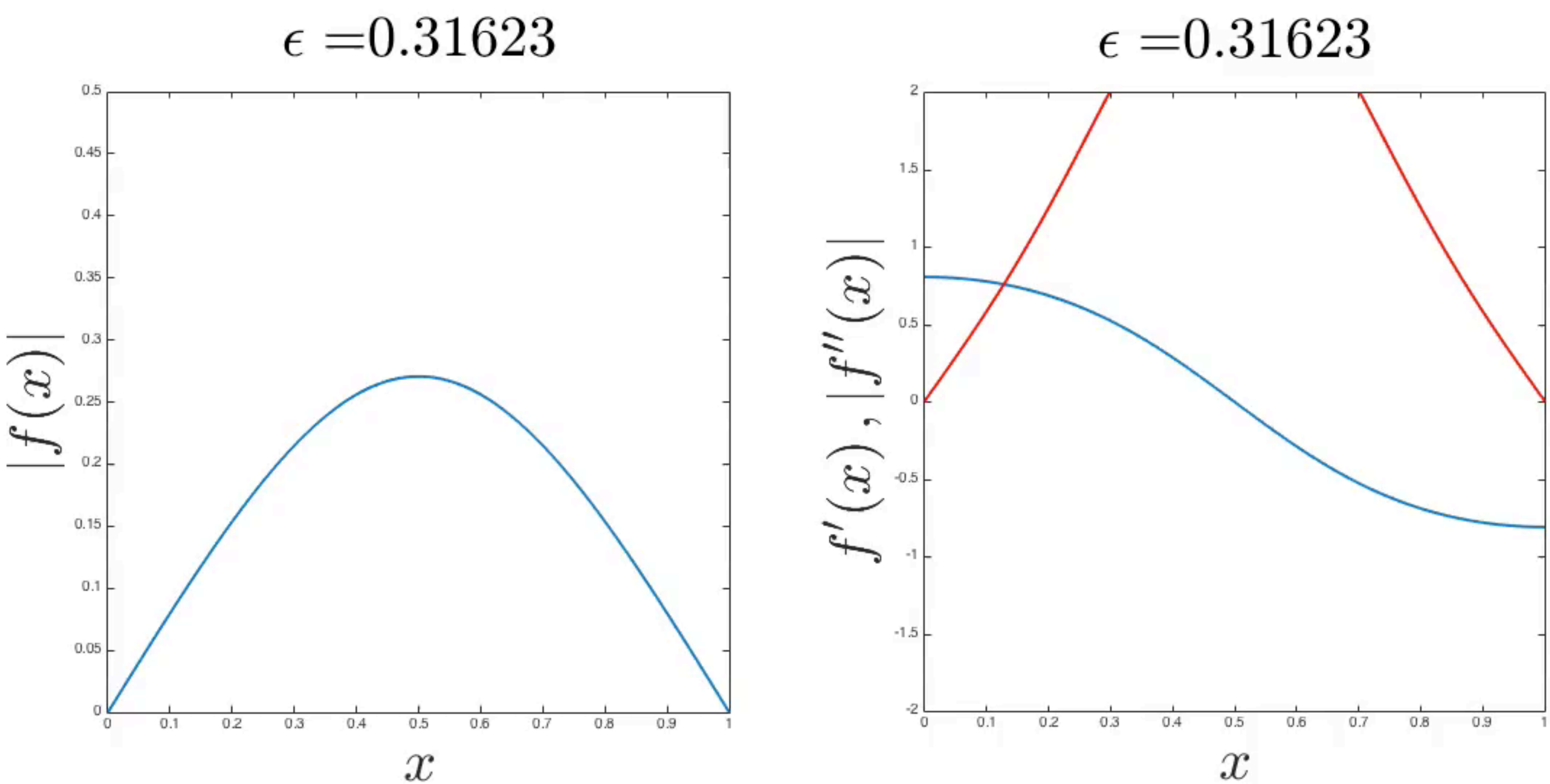
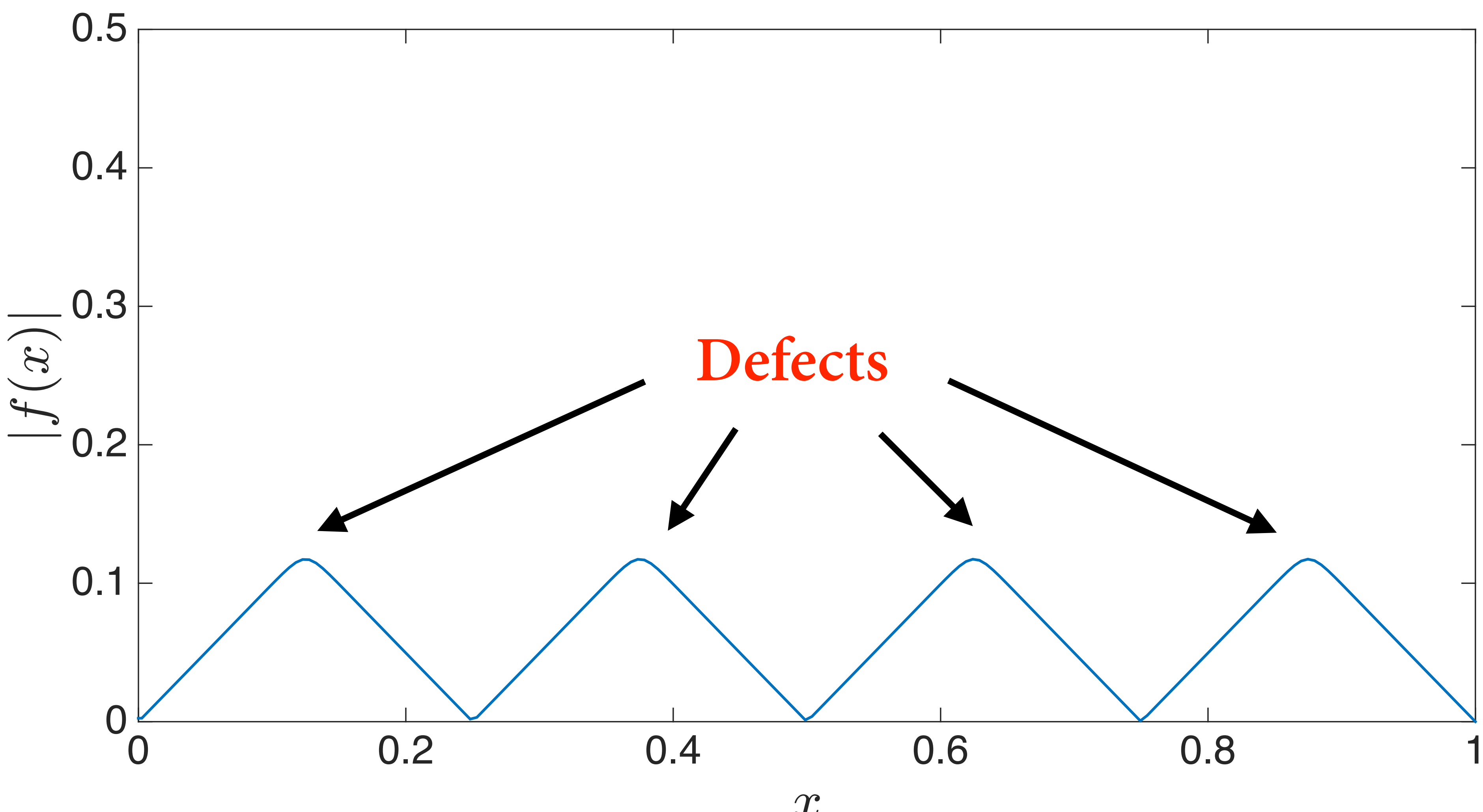


Energy concentrates on tip of triangle (Defect).

Calculus of Variations Example 4

$$E[f] = \delta^2 \int_0^1 f(x)^2 \, dx + \int_0^1 \left(\left(\frac{df}{dx} \right)^2 - 1 \right)^2 dx + \epsilon^2 \int_0^1 \left(\frac{d^2 f}{dx^2} \right)^2 dx.$$

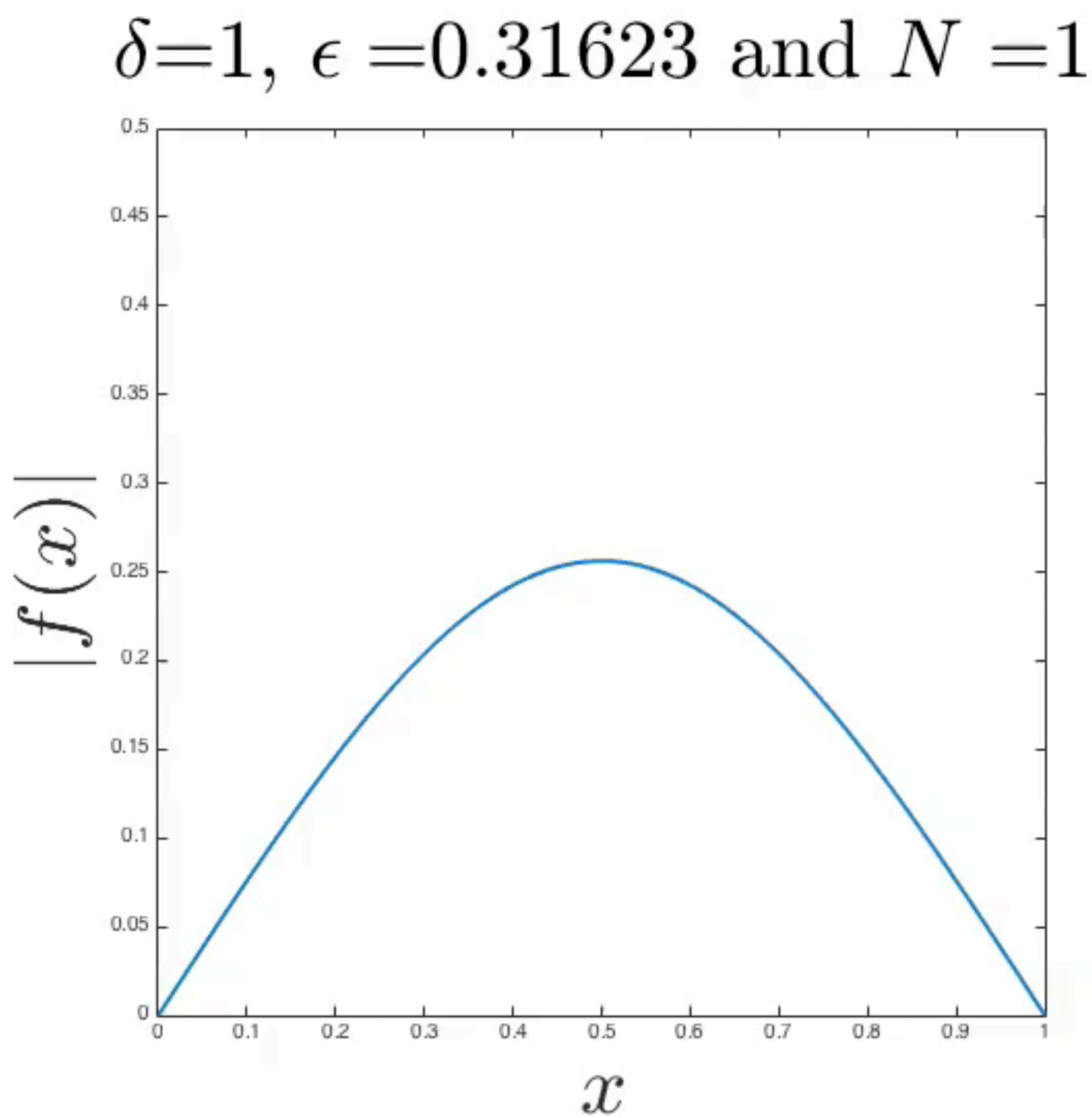


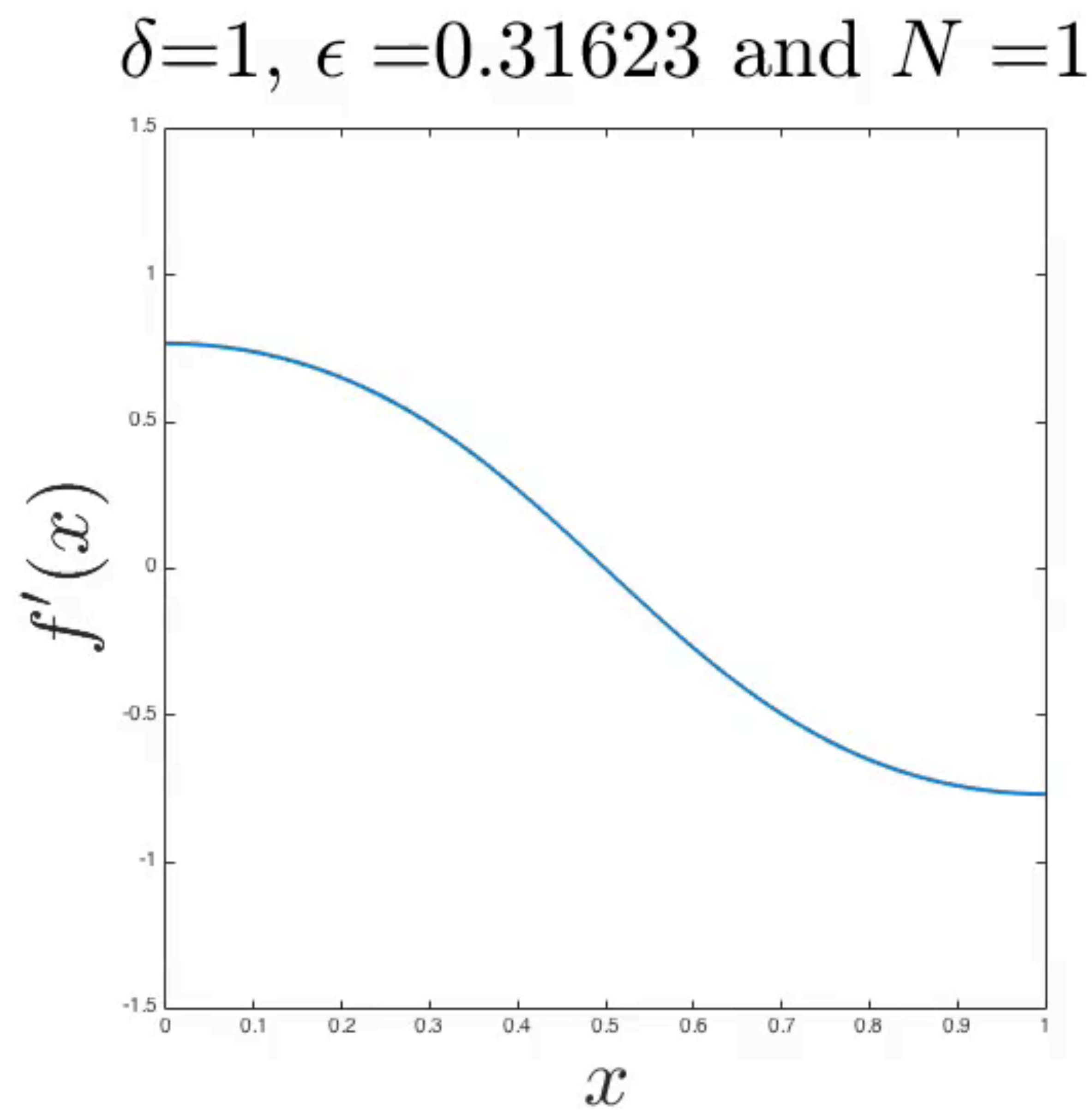
$$N = \#\text{ of triangles} \Rightarrow E[f] \approx \frac{1}{8N^2} + \frac{8}{3}\epsilon N$$

$$\text{Optimizing} \Rightarrow N = .45\delta^{\frac{2}{3}}\epsilon^{-\frac{1}{3}}$$

Calculus of Variations Example 4

$$E[f] = \delta^2 \int_0^1 f(x)^2 \, dx + \int_0^1 \left(\left(\frac{df}{dx} \right)^2 - 1 \right)^2 dx + \epsilon^2 \int_0^1 \left(\frac{d^2 f}{dx^2} \right)^2 dx.$$





Energy concentrates on defects.

Gamma-Limits

Definition: Let $F_n : X \mapsto \mathbb{R}$. We say F_n **Γ-converges** to $F : X \mapsto \mathbb{R}$ and write $\Gamma - \lim_{n\to\infty} F_n = F$ if

1. **Asymptotic Lower Bound:** For every $x \in X$, and every $x_n \to x$ in X:
$$F(x) \leq \liminf_{n\to\infty} F_n(x_n).$$
2. **Recovery Sequence:** For every $x \in X$, there exists some $x_n \in X$ such that $x_n \to X$ in X and:
$$F(x) \geq \limsup_{n\to\infty} F_n(x_n).$$

Theorem (Fundamental Theorem of Γ-Convergence)

If $F_n \xrightarrow{\Gamma} F$ and x_n minimizes F_n, then every limit point of the sequence $(x_n)_{n\in\mathbb{N}}$ is a minimizer of F.

Examples:

$$\Gamma - \lim_{n\to\infty} \cos(nx)?$$

$$\Gamma - \lim_{n\to\infty} \left(\frac{1}{2}x^2 + \sin(nx)\right)?$$

$$\Gamma - \lim_{n\to\infty} \begin{cases} 0 & x \leq -\frac{1}{n} \\ \frac{nx+1}{2} & -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 1 & x \geq \frac{1}{n} \end{cases}?$$

$$\Gamma - \lim_{n\to\infty} nx\exp(nx)?$$

Calculus of Variations Example 3

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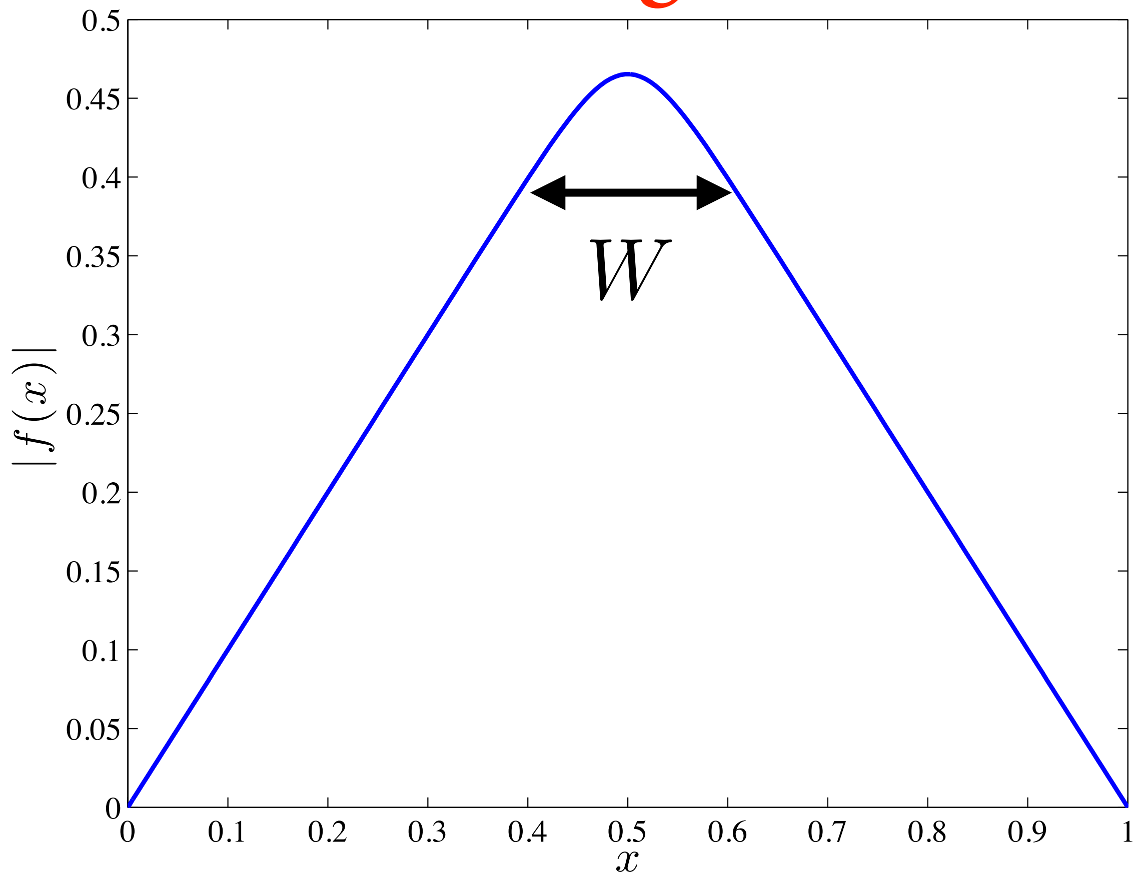
Good guess:



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Observations

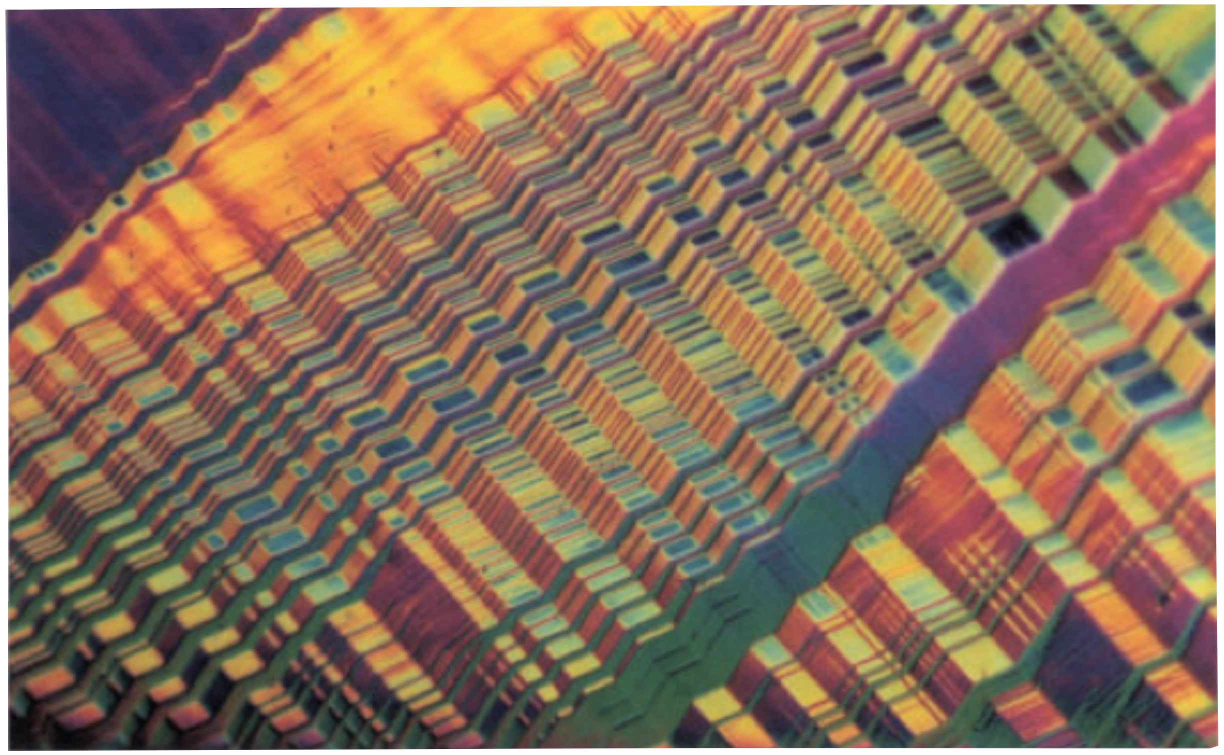
$$E[f] = \delta^2 \int_0^1 f(x)^2\,dx + \int_0^1 \left(\left(\frac{df}{dx} \right)^2 - 1 \right)^2 dx + \epsilon^2 \int_0^1 \left(\frac{d^2 f}{dx^2} \right)^2 dx.$$

1. **Competition** between different terms energy scales drives pattern formation.
2. **"Simple" differential equations** give the correct pattern.
3. Knowing how the **patterns partition energy (scaling laws)**, the problem reduces to solving a **discrete optimization problem**.

Big Picture:

- In an indirect manner we have developed **rigorous asymptotic approximations** to solutions to a fourth order nonlinear differential equation.
- We have also developed a simple **robust discrete numerical algorithm** for solving the equation.
- This technique is similar to the **viscosity method** for regularizing shocks in hyperbolic systems.

Why Care?



Live Update on Rare Events

Snowing in Tucson!