Chapter 8: Bifurcations Part Deux

\[ \begin{align*}
\dot{x} &= f(x, y, \mu) \\
\dot{y} &= g(x, y, \mu)
\end{align*} \]

A bifurcation point \( \mathbf{x}^* \) is a point where the topology of the phase portrait changes.

1. Let \( (x^*, y^*) \) denote an equilibrium point.
2. Let \( \lambda_1, \lambda_2 \) denote the eigenvalues associated with \( \mathbf{x}^* \).

Bifurcations occur if one or both of the eigenvalues \( \lambda_1, \lambda_2 \) lie on the imaginary axis.

1. \( \lambda_1 = \pm i \omega \) \( \implies \) Hopf bifurcation (new stuff)
2. \( \lambda_1 = 0, \lambda_2 \neq 0 \) \( \implies \) 1-D bifurcation
3. \( \lambda_1, \lambda_2 = 0 \) \( \implies \) \( J = (0 1)^T \), never really happens

1-D Bifurcations

1. Saddle Node:
\[ \begin{align*}
\dot{x} &= \mu - x^2 \\
\dot{y} &= -y
\end{align*} \]

More Generally: One nullcline slips through another.

Null-clines are wrong, should be vertical.

More specifically, one nullcline intersects another.

\( N < N_0 \) \quad \text{(blue nullcline intersects pink one)} \}
2. Transcritical\n\[
\begin{align*}
  \dot{x} &= \nu x - x^2 \\
  \dot{y} &= -y
\end{align*}
\]

*Fixed point at origin switches stability.

3. Pitch fork\n\[
\begin{align*}
  \dot{x} &= \nu x \pm x^3 \\
  \dot{y} &= -y
\end{align*}
\]

Take - sign \rightarrow

Example:\n\[
\begin{align*}
  S &= r_s S \left(1 - \frac{S}{K_s} \frac{k_E}{E}\right) \\
  E &= r_E E \left(1 - \frac{E}{k_E}\right) - \frac{PB}{S}
\end{align*}
\]

\[S \sim \text{size of the forest}\]
\[B \sim \text{worm population}\]
\[K_s \sim \text{spruce tree carrying capacity when } E = k_E\]
\[k_E \sim \text{energy reserve carrying capacity}\]
\[B \sim \text{budworm population}\]
\[P \sim \text{rate energy reserve is eaten by worms}\]
\[r_s \sim \text{growth rate of forest}\]
\[k_E \sim \text{growth rate of energy reserves}\]
Rescale

\[ x = S / k_s \]
\[ y = E / k_E \]
\[ \tau = r_s t \]

\[ \Rightarrow \frac{dx}{d\tau} = x \left(1 - \frac{x}{\beta} \right) \]
\[ \frac{dy}{d\tau} = xy(1-y) - \frac{\beta}{x} \]

\[ \alpha = \frac{r_E}{r_s} \]
\[ \beta = \frac{P_B}{k_B r_s} \]

**Null-clines**

\[ \frac{dx}{d\tau} = 0, \quad x = 0, \quad y = x \]

\[ \frac{dx}{d\tau} = 0, \quad x = \frac{B}{\alpha y(1-y)} \]

Let's sketch the null-clines for this system. First, let's figure out what the non-trivial null-cline looks like.
Therefore, we have two types of phase portraits:

Ignored corner

\[ \begin{array}{c}
\text{Small } \beta, \\
\rightarrow \text{Forest can live with tree beds}
\end{array} \]

\[ \begin{array}{c}
\text{Large } \beta, \\
\rightarrow \text{Forest dies}
\end{array} \]

How can we determine bifurcation point? Too hard...
However, the bifurcation is a saddle-node.
Hoopf-Bifurcation

Let \( w \neq 0 \)

\[
\begin{align*}
\dot{x} &= wx - wy \pm (x^3 + x^2) + b(-x^2y - y^3) \\
\dot{y} &= wx + wy \pm (x^3 + y^3) + b(x^3 + x^2y)
\end{align*}
\]

linear rotation cubic nonlinearity

\( J(0,0) = \begin{pmatrix} w & -w \\ w & w \end{pmatrix} \), eigenvalues \( \lambda_1, \lambda_2 = w \pm iw \)

Convert to polar coordinates:

\[
\begin{align*}
\dot{r} &= wr \pm r^3 \\
\dot{\theta} &= w + br^2
\end{align*}
\]

1. **Supercritical Hopf-bifurcation for \( "-" \) sign**

   Fixed points:

   \( r = 0, \ r = \sqrt{w} \)

2. **Subcritical Hopf-bifurcation for \( "+" \) sign**

   Fixed points:

   \( r = 0, \ r = \sqrt{-w} \)
Example:

"Chemical Reaction"

\[
\begin{align*}
\dot{x} &= a - x + x^2 y \\
y &= b - x^2 y \\
\end{align*}
\]

All wrong!!

Nullclines:

\[
\begin{align*}
\frac{dx}{dt} &= 0, \\
y &= \frac{x-a}{x^2} \\
\frac{dy}{dt} &= 0, \\
x &= \frac{b}{x^2}
\end{align*}
\]

Ignore!!

Crazy wrong!!
Example

Chemical reaction

\[ x = a - x + x^2 \]
\[ y = b - x^2 y \]

Null-clines

\[ \frac{dx}{dt} = 0 \Rightarrow y = x - a \]
\[ \frac{dy}{dt} = 0 \Rightarrow y = \frac{b}{x^2} \]

Fixed Point

\( (a+b, \frac{b}{a+b}^2) \)

\[ \frac{dy}{dx} \approx -1 \text{ for large } x \]

Solve

\[ \frac{dy}{dx} < -1 \Rightarrow \frac{b - x^2 y}{a - x + x^2 y} < -1 \Rightarrow x > b + a \]

Solve

\[ \frac{b - x^2 y}{a - x + x^2 y} < \frac{b}{a} \Rightarrow y > \frac{1}{x(a+1)} \]
Since we have a tripping region we now check stability of the fixed point.

\[
J = \begin{pmatrix} -1 + 2xy & x^2 \\ -2xy & -x^2 \end{pmatrix}
\]

\[
J(a + b, \frac{b}{(a + b)^2}) = \begin{pmatrix} -1 + \frac{2b}{a + b} (a + b)^2 \\ -2 \frac{b^2}{a + b} - (a + b)^2 \end{pmatrix}
\]

Hopf bifurcation occurs when

\[
\text{Tr}[J(a + b, \frac{b}{(a + b)^2})] = 0
\]

\[
\Rightarrow -1 + 2 \frac{b}{a + b} - (a + b)^2 = 0
\]

\[
\Rightarrow b - a = (a + b)^3
\]

This is a super-critical Hopf bifurcation. *Really need a numerical scheme to understand complete dynamics.*

The period can be estimated by assuming a circular orbit at the bifurcation point.

\[
\det(J(a + b, \frac{b}{(a + b)^2})) = (a + b)^2 - 2b(a + b) + 2b(a + b) = -\lambda_1^2
\]

at the bifurcation point.

\[
\Rightarrow \lambda_1^2 = b(a + b).
\]

The period is then

\[
T = \frac{2\pi}{a + b}
\]

Since the linear solution is:

\[
x = A \cos(\lambda_1 t)
\]

\[
y = A \sin(\lambda_1 t)
\]
Other Periodic Bifurcations

1. Saddle Node:
\[ \dot{r} = \mu r - r^3 + r^5 \rightarrow \text{Quintic normal form} \]

\[ N \geq \frac{1}{4} \]

\[ N > \frac{1}{4} \]

2. Infinite Periodic:
\[ \dot{\theta} = r(1-r^2) \]
\[ \dot{\theta} = \mu - \delta \sin \theta \rightarrow \text{Saddle node bifurcation} \]

\[ N < \frac{1}{4} \]

\[ N > 1 \]

"pink curves not correct."
3. Homeoclinic Bifurcation

Example (Forced Pendulum with Friction)
\[ \ddot{\phi} + \alpha \dot{\phi} + \sin(\phi) = I, \quad I, \alpha > 0. \]
\[ \phi \in \mathbb{S}^1 \]

Let \( \mathbf{v} = \phi \) then
\[ \dot{\mathbf{v}} = I - \alpha \mathbf{v} - \sin(\phi) \]
\[ \phi = \mathbf{v} \]

Fixed points only exist if \( I < 1, \mathbf{v} = 0, \sin(\phi) = I \)

\[
\begin{bmatrix}
0 & 1 \\
-\cos(\phi) & -\alpha
\end{bmatrix}
\Rightarrow \text{The eigenvalues are}
\begin{align*}
\lambda_1, \lambda_2 &= -\alpha \pm \sqrt{\alpha^2 - 4 \cos^2(\phi)} \\
&= -\alpha \pm \sqrt{\alpha^2 - 4 (1 - \frac{I^2}{\alpha^2})}
\end{align*}
\]

\( \Rightarrow \) 1 fixed point is a saddle, the other a stable fixed point of some type.
Analogous with Josephson junction

\[ \phi = \phi_1 - \phi_2 \rightarrow \text{phase difference} \]

If \( I < I_c \), the junction acts as if it had zero resistance. There is a phase difference between the states.

If \( I > I_c \), the junction acts as a resistor. The voltage is given by

\[ V = \frac{\hbar}{2e} \langle \phi \rangle \]

Case 1: \( I > I_c \)

\[ \phi = \nabla \]

\[ V = I - \alpha \nabla - \sin(\phi) \]

Nullcline

\[ V = 0 \]

\[ V = \frac{1}{\alpha} (I - \sin(\phi)) \]

The Poincaré map \( P : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) maps initial \( V \) at \( \phi = 0 \) to value of \( V \) at \( \phi = 2\pi \) for a solution trajectory.

1. \( P(V_1) < V_1 \)
2. \( P(V_2) > V_2 \)
Let's sketch $P(v)$.

$p(v)$ must be monotonically non-decreasing so trajectories could cross.

$\Rightarrow \exists v^* \text{ such that } P(v^*) = v^*$. This implies the existence of a limit cycle.

Case 2:

$I < I_c, \alpha << 1$

$\Rightarrow$ saddle fixed point

$2\pi \Rightarrow$ limit cycle still exists, however, we also have a stable fixed point.
Case 3:
$I = I_c < 1, \alpha \ll 1$.
Limit cycle merges with stable manifold.

Infinite period bifurcation
Homoclinic bifurcation

Case 4:
$\alpha \gg 1, I < 1$.

No limit cycle, Pendulum dies down

Voltage of limit cycle.
Hysteresis.
Voltage of spiral.
Motion on Surfaces

Differential geometry in a nutshell → Cashew

Vector field on hyperbolid

Image of vector field on a coordinate chart.

All math is done on coordinate chart and then is mapped back to surface (Manifold) use rules of differential geometry.

Examples:

1. Glue sides together to form a cylinder.

2. Glue opposite sides together to form a torus.
= Möbius band

= Klein bottle.

It exists in my brain so it exists - Descartes.

Other Crazy Surfaces
\[ \mathbb{RP}^2 = \text{set of all lines through the origin in } \mathbb{R}^3 \]

We only need to sew antipodal points on equator.

This is a Möbius band sewed to the equator of a disc.

Motion on a torus.

\[ \dot{\theta}_1 = \omega_1 + k_1 \sin (\theta_2 - \theta_1) \]
\[ \dot{\theta}_2 = \omega_1 + k_2 \sin (\theta_1 - \theta_2) \]

Crazy periodic orbit!!
Let \( t = t_1 - t_2 \)

\[ t = w - k \sin(t) \]

\[ w = w_1 - w_2 \]

\[ k = k_1 + k_2 \]

If \( \frac{w_1}{k_1} < 1 \) two fixed points where \( \sin(t^*) = \frac{w_1}{k_1} \)

\[ \dot{t} = w_1 - k_1 \left( \frac{w_1 - w_2}{k_1 + k_2} \right) = \frac{k_2 w_1 + k_1 w_2}{k_1 + k_2} \]

\[ \dot{t}_1 = \frac{k_2 w_1 + k_1 w_2}{k_1 + k_2} \]

\[ \Rightarrow \frac{d\dot{t}_1}{d\dot{t}} = 1 \Rightarrow A + \text{equilibrium} \]

\[ \text{Slope 1. Attracting fixed point.} \]

\[ \text{Slope 1. Repelling fixed point.} \]

**Uncoupled System**

\[ \dot{\theta}_1 = w_1 \]

\[ \dot{\theta}_2 = w_2 \]

\[ \frac{d\dot{\theta}_1}{d\dot{\theta}} = \frac{w_2}{w_1} \rightarrow \text{slope} \]

If \( \frac{w_1}{w_2} \) is rational, \( p, q \) such that \( \frac{w_1}{w_2} = \frac{p}{q} \)

\[ \Rightarrow \text{When } \theta_2 \text{ completes } p \text{ revolutions } \theta_1 \text{ completes } q \text{ revolutions.} \]
Example:
\[
\frac{d\theta_2}{d\theta_1} = \frac{3}{4}
\]

Example:
\[
\frac{d\theta_2}{d\theta_1} = \sqrt{2}
\]

"quasi-periodic → trajectories never close."