Dual Space of ℓ^p

We want to prove that that the dual space of ℓ^p is ℓ^q. We do this in parts.

1. Let $L \in B(\ell^p, \mathbb{R})$. Then, for $a = (a_1, a_2, \ldots) \in \ell^p$ it follows that

$$\begin{aligned} L(a) &= L((a_1, a_2, \ldots)) \\ &= L((a_1, 0, \ldots)) + L(0, a_2, 0, \ldots) + \ldots \\ &= a_1 L((1,0,0,\ldots)) + a_2 L((0,1,0,\ldots)) + \ldots \\ &= a_1 \beta_1 + a_2 \beta_2 + \ldots, \end{aligned}$$

where

$$\beta_1 = L((1,0,0,\ldots)),\ \beta_2 = L((0,1,0,\ldots)), \ldots.$$

Therefore, for each $L \in B(\ell^p, \mathbb{R})$ there exists a sequence $\beta = (\beta_1, \beta_2, \beta_3, \ldots)$ corresponding to L. Moreover,

$$L(a) = \sum_{i=1}^{\infty} a_i \beta_i$$

2. By Hölders inequality it follows that

$$|L(a)| \le \sum_{i=1}^{\infty} |a_i \beta_i| \le \|a\|_{\ell^p} \|\beta\|_{\ell^q}$$

Consequently, if $\beta \in \ell^q$ then L is bounded.

3. Now, suppose L is bounded. Let

$$a^{(n)} = (\operatorname{sgn}(\beta_1)|\beta_1|^{q-1}, \operatorname{sgn}(\beta_2)|\beta_2|^{q-1}, \ldots, \operatorname{sgn}(\beta_n)|\beta_n|^{q-1}, 0, 0, \ldots)$$

Then

$$\frac{|L(a^{(n)})|}{\|a\|_p} = \frac{\sum_{i=1}^{n} |\beta_i|^q}{\left(\sum_{i=1}^{n} |\beta_i|^{p(q-1)}\right)^{1/p}} = \frac{\sum_{i=1}^{n} |\beta_i|^q}{\left(\sum_{i=1}^{n} |\beta_i|^{q}\right)^{1/p}} = \left(\sum_{i=1}^{n} |\beta_i|^q\right)^{1/q}$$

Therefore,

$$\sup_n \frac{|L(a^{(n)})|}{\|a\|_p} = \lim_{n\to\infty} \left(\sum_{i=1}^{n} |\beta_i|^q\right)^{1/q} = \|\beta\|_{\ell^q}.$$

Consequently, since L is bounded it follows that:

$$\infty > \sup_a \frac{|L(a)|}{\|a\|_p} > \sup_n \frac{|L(a^{(n)})|}{\|a\|_p} = \|\beta\|_{\ell^q}.$$

Therefore, if L is bounded then $\beta \in \ell^q$.

4. Finally, it follows from items 2 and 3 that

$$\|L\|_{op} = \|\beta\|_{\ell^q}.$$

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