Analysis
Spring 2018
Exam 2
03/??/18

Name (Print): Key

This exam contains 9 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work,** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Short answer questions:** Questions labeled as **"Short Answer"** can be answered by simply writing an equation or a sentence or appropriately drawing a figure. No calculations are necessary or expected for these problems.
- **Unless the question is specified as short answer, mysterious or unsupported answers might not receive full credit.** An incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

1. (25 points) Do either part (a) or part (b). Circle which problem you want graded.

   (a) (25 points) Consider the following space of continuously differentiable functions on $[-1,1]$ which vanish at the endpoints:

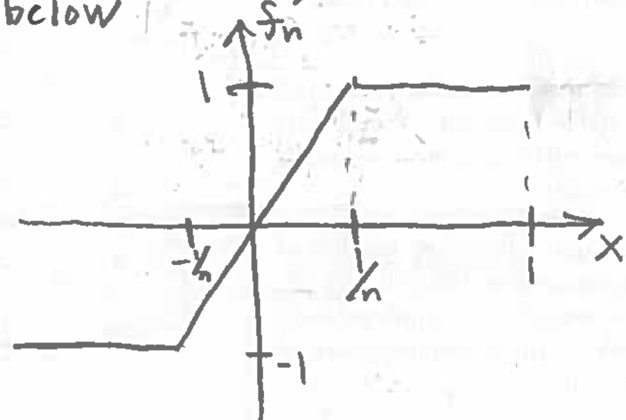
$$C_0^1([-1,1]) = \{f : [-1,1] \mapsto \mathbb{R} : f, f' \in C([-1,1]) \text{ and } f(-1) = f(1) = 0\}.$$

Is this space complete with respect to the following norm:

$$\|f\|_{W_0^{1,1}} = \int_{-1}^{1} |f'(x)|\, dx?$$

Prove or provide a counterexample. If you provide a counterexample it can be pictorial.

No. Consider the sequence of functions with f_n' plotted below



Clearly, $f_n' \to \begin{cases} 1, & x > 0. \\ 0, & x \leq 0 \end{cases}$

(b) (25 points) Consider the sequence of functions $f_n(x)$ defined on the interval $[0,1]$ by

$$f_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k^2}.$$

- Prove that f_n is Cauchy with respect to the sup norm: $\|\cdot\|_\infty$.
- Prove that

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^2}$$

is a continuous function.

$$|f_m(x) - f_n(x)| = \left|\sum_{k=1}^{n} \frac{\sin(kx)}{k^2} - \sum_{k=1}^{m} \frac{\sin(kx)}{k^2}\right|$$

$$= \left|\sum_{k=n}^{m} \frac{\sin(kx)}{k^2}\right|$$

$$\leq \sum_{k=n}^{m} \frac{|\sin(kx)|}{k^2}$$

$$\leq \sum_{k=n}^{m} \frac{1}{k^2}$$

$$\leq \sum_{k=n}^{\infty} \frac{1}{k^2}.$$

$$\Rightarrow \|f_m - f_n\|_\infty \leq \sum_{k=n}^{\infty} \frac{1}{k^2},$$

which implies f_m is Cauchy. Since $C([0,1])$ is complete it follows with respect to $\|\cdot\|_\infty$ it follows that

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^2}$$

is continuous.

2. (25 points) Do either part (a) or part (b). Circle which problem you want graded.

   (a) (25 points) Are the L^∞ and L^1 metrics on $C([-1,1])$ equivalent? Prove or provide a counterexample.

No. Consider the following sequence:

$$f_n(x) = \begin{cases} 1, & -\frac{1}{n} \le x \le \frac{1}{n} \\ 0, & \text{o.w.} \end{cases}$$

Clearly,

$$\|f_n\|_{L^1} = \frac{2}{n} \to 0.$$

$$\|f_n\|_{L^\infty} = 1.$$

(b) (25 points) f is a continuously differentiable function defined on $[0,1]$ such that $f(0) = f(1) = 0$. Prove that

$$\|f\|_{L^p} \le \|f'\|_{L^p}.$$

$$f(x) = \int_0^x f'(t)\,dt$$

$$\Rightarrow |f(x)| \le \int_0^x |f'(t)|\,dt$$

$$\le \int_0^1 |f'(t)|\,dt$$

$$\le \left(\int_0^1 |f'(t)|^p\,dt\right)^{1/p} \left(\int_0^1 1^q\,dt\right)^{1/q}$$

$$= \|f'\|_{L^p}$$

$$\Rightarrow \int_0^1 |f(x)|^p\,dx \le \int_0^1 \|f'\|_{L^p}^p\,dx$$

$$= \|f'\|_{L^p}^p$$

$$\Rightarrow \|f\|_{L^p} \le \|f'\|_{L^p}.$$

3. (25 points) Do either part (a) or part (b). Circle which problem you want graded.

   (a) (25 points) A function $f \in C([0,1])$ is called Hölder continuous of exponent $\alpha > 0$ if the quantity

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite. Let

$$A = \{f \in C([0,1] : \|f\|_\infty \leq 1 \text{ and } N_\alpha(f) \leq 1\}.$$

Is the closure of A compact with respect to the norm $\|\cdot\|_{L^\infty}$? Prove or provide a counterexample.

Let $f_n \in A$. Therefore,

$$|f_n(x) - f_n(y)| \leq |x-y|^\alpha$$

and consequently f_n is equicontinuous. Since f_n is bounded it follows from Arzela-Ascoli that f_n has a convergent subsequence.

(b) (25 points) Use Gronwall's inequality to prove that solutions to the following initial value problem are unique:

$$\frac{du}{dt} = |\sin(u)| + 2,$$
$$u(0) = u_0.$$

Let u, v be solutions. Then,

$$\left|\frac{du}{dt} - \frac{dv}{dt}\right| = \big||\sin(u)| - |\sin(v)|\big|$$
$$u(0) - v(0) = 0.$$

By the Mean-Value Theorem:

$$\frac{\big||\sin(u)| - |\sin(v)|\big|}{|u-v|} = |\cos(c)| \leq 1$$

$$\Rightarrow \big||\sin(u)| - |\sin(v)|\big| \leq |u-v|.$$

Therefore,

$$\left|\frac{du}{dt} - \frac{dv}{dt}\right| \leq |u-v|$$
$$u(0) - v(0)$$

It follows from Gronwall's inequality that

$$|u(t) - v(t)| \leq 0e^{-t} = 0.$$

4. (25 points) Do either part (a) or part (b). Circle which problem you want graded.

(a) (25 points) Let $L > 0$ and assume the following initial-boundary value problem

$$\begin{aligned} u_t &= -u - u_{xxxx}, \\ u(x,0) &= u_0(x), \\ u(0,t) &= u(L,t) = 0, \\ u_x(0,t) &= u_x(L,t) = 0, \end{aligned}$$

has a smooth solution $u(x,t)$ for all smooth initial conditions $u_0(x)$. Prove that smooth solutions to this equation are unique.

Let u solve this I.B.V.P. Calculating it follows that

$$\begin{aligned} \frac{d}{dt}\|u\|_{L^2}^2 &= \int_0^L 2u \cdot u_t \, dx \\ &= \int_0^L 2u(-u - u_{xxxx})\,dx \\ &= \int_0^L (-2u^2 - 2u \cdot u_{xxxx})\,dx \\ &= \int_0^L -2u^2 dx - 2\cancel{u u_{xxx}}\Big|_0^L + 2\int_0^L u_x u_{xxx}\, dx \\ &= \int_0^L -2u^2 dx + 2\cancel{u_x u_{xx}}\Big|_0^L - 2\int_0^L u_{xx}^2\, dx \\ &= -2\left(\int_0^L u^2 dx + \int_0^L u_{xx}^2\, dx\right) \\ &\le 0. \end{aligned}$$

Now, if u, v both solve this I.B.V.P. then

$$\|u(x,0) - v(x,0)\|_{L^2} = \|u_0(x) - v_0(x)\|_{L^2} = 0$$

$$\Longrightarrow \|u(x,t) - v(x,t)\|_{L^2} = 0.$$

(b) (25 points) Suppose that (X, d) is a complete metric space. Prove that (X, d) and $(\bar{X}, \bar{d})$ are isometric. That is, find a function $F : X \mapsto \bar{X}$ that is an isometry and prove that F is injective and surjective.

Let $F : X \to \tilde{X}$ be defined by

$$F(x) = [x].$$

Clearly, F is one-to-one. Let $y \in \tilde{X}$. Then, there exists a Cauchy sequence $x_n \in X$ such that

$$y = [x_n].$$

Since X is complete there exists $z \in X$ such that $x_n \to z$. Consequently, $[x_n] \sim [z]$. Therefore,

$$F(z) = y.$$

Finally, if $a, b \in \tilde{X}$ then there exist $c, d \in X$ such that

$$F(c) = [c] = a,$$
$$F(d) = [d] = b$$

Consequently,

$$\begin{aligned}\tilde{d}(a,b) &= \tilde{d}(F(c), F(d)) \\ &= \lim_{n\to\infty} d(c,d) \\ &= d(c,d).\end{aligned}$$