# Homework 11

Analysis

Due: April 23, 2018

1. Let $X, Y$ be normed linear spaces. Prove that if $Y$ is complete the $B(X, Y)$ is a complete space with respect to the operator norm.

2. **Fourier Series:** With every function $f \in C([0,1])$ we can associate a sequence $a_n$ by

$$f(x) \mapsto a_n = \int_0^1 f(x) \sin(2\pi n x)\, dx.$$

The series $a_n$ is called the Fourier sine series of $g$, and we will denote the map from $C([0,1])$ to sequences by $\mathcal{F}$.

   (a) Show that $\mathcal{F}$ is a continuous mapping between $(C([0,1]), \|\cdot\|_{L^1})$ and $l^\infty$.

   (b) Show that $\mathcal{F}$ is a continuous mapping between $(C([0,1]), \|\cdot\|_{L^2})$ and $l^2$.

3. $l_c$ is the space of all real valued sequences that have only a finite number of non-zero terms. Recall that $c_0$ is the space of all sequences $a_n \in \mathbb{R}$ satisfying $\lim_{n\to\infty} a_n = 0$.

   (a) If $c_0$ is equipped with the $\|\cdot\|_\infty$ norm, what is the dual space of $c_0$? That is, what is the space of bounded linear functionals?

   (b) Show that $l_c$ is a vector space over $\mathbb{R}$.

   (c) Show that $l_c$ is dense in the sequence space $l^p$ with respect to the $l^p$ norm.

   (d) Show that the closure of $l_c$ in the sup norm is $c_0$.

4. Consider the mapping defined by

$$f(x) \mapsto Tf(x) = \int_0^\pi \sin(x-y) f(y)\, dy.$$

   (a) Show that $T$ maps functions in $C([0,\pi])$ into $C([0,\pi])$.

   (b) Show that $T$ is a continuous mapping from $(C([0,1]), \|\cdot\|_{L^1})$ into $(C([0,1]), \|\cdot\|_{L^1})$.

   (c) Show that $T$ is a continuous mapping from $(C([0,1]), \|\cdot\|_{L^\infty})$ into $(C([0,1]), \|\cdot\|_{L^\infty})$.

5. Let $\mathcal{A} = \{u \in C^1([a,b]) : u(a) = \alpha \text{ and } u(b) = \beta\}$. Consider the functional $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[u] = \int_a^b g(x,u) \sqrt{1 + \left(\frac{du}{dx}\right)^2}\, dx,$$

where $g$ is smooth function.

   (a) Calculate the *weak form* of the Euler-Lagrange equations for this functional.

   (b) Calculate the *strong form* of the Euler-Lagrange equations for this functional.

6. Let $\mathcal{A} = \{u \in C^1([-1,1]) : u(-1) = 0 \text{ and } u(1) = 1\}$. Consider the functional $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[u] = \int_{-1}^1 u(x)^2 \left(1 - \frac{du}{dx}\right)^2 dx.$$

Prove that $I$ does not have a minimizer in $\mathcal{A}$.