Homework 12

Analysis

Due: May 02, 2018

1. Prove that equivalent norms on a normed linear space X lead to equivalent norms on the space $B(X, X)$ of bounded linear operators on X.

2. Let $x^{(n)} = (0, \ldots, 0, 1, 0 \ldots) \in l^2$ which has a 1 in the $n^{(th)}$ entry and zeros elsewhere. Prove that $x^{(n)} \rightharpoonup 0$.

3. Does the sequence of functions $f_n(x) = x^n$ converge weakly to zero in $L^2([0,1])$. If so prove it. If not prove it.

4. **Types of weak convergence**.
   (a) Let $f \in L^2(\mathbb{R})$ be a nonzero function and define a sequence of functions $f_n(x)$ by $f_n(x) = f(x-n)$. Prove that $f_n \rightharpoonup 0$ but f_n does not converge strongly in $L^2(\mathbb{R})$.
   (b) Prove that the sequence $f_n(x) = \sqrt{2}\sin(n\pi x)$ converges weakly to zero in $L^2([0,1])$. Prove that f_n does not converge strongly. **Hint:** The ideas you used in problem 2(b) from the last homework might help.
   (c) Let $f \in L^2(\mathbb{R})$ be a nonzero function and define a sequence of functions $f_n(x)$ by $f_n(x) = \sqrt{n} f(nx)$. Prove that $f_n \rightharpoonup 0$ but f_n does not converge strongly.

5. Let $f_n \in L^2([0,1])$ be a countably infinite sequence satisfying

$$\begin{cases} \displaystyle\int_0^1 f_n(x) f_m(x)\, dx = 1 & \text{if } m = n \\ \displaystyle\int_0^1 f_n(x) f_m(x)\, dx = 0 & \text{if } m \neq n \end{cases}.$$

   Prove that $f_n \rightharpoonup 0$ in $L^2([0,1])$.

6. Let $x^{(m,n)} \in l^2$ be the sequence whose m^{th} entry is 1, n^{th} entry is m, and all other entries are zero. Let $A \subset l^2$ be the set of all such sequences. Show that no sequence of elements in A converges weakly to 0.