Homework 2

Analysis

Due: January 29, 2018

1. **Metric Spaces**:

   (a) Prove that $(\mathbb{R}, d)$ is a metric space with metric $d(x, y) = |x - y|$.

   (b) Prove that $(\mathbb{R}^n, d_1)$ is a metric space with metric

   $$d_1(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |y_i - x_i|.$$

   (c) Prove that $(\mathbb{R}^n, d_\infty)$ is a metric space with metric

   $$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{i \in \{1, \ldots n\}} |y_i - x_i|.$$

   (d) Let X be the set of n-letter words in a k character alphabet $A = \{a_1, a_2, \ldots, a_k\}$, meaning that $X = \{(x_1, x_2, \ldots x_n) : x_i \in A\}$. We define the distance between two words $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ to be the number of places in which the words have different letters. That is,

   $$d(x, y) = \#\{i : x_i \neq y_i\}.$$

   Prove that (X, d) is a metric space.

2. **Discrete Metric**:

   (a) Prove that $(\mathbb{R}^n, d)$ is a metric space with metric

   $$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}.$$

   This metric space is called the **discrete metric**.

   (b) Show that in this metric space topology every subset of $\mathbb{R}^n$ is open.

   (c) Show that in this metric space topology a sequence x_n converges to $x \in \mathbb{R}^n$ if and only if there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n = x$.

3. **Building Metric Spaces:** (X, d_X) and (Y, d_Y) are metric spaces.

   (a) If $U \subseteq X$, show that (U, d_U) is a metric space where $d_U(u_1, u_2) = d_X(u_1, u_2)$ for all $u_1, u_2 \in U$.

   (b) Show that the Cartesian product $X \times Y$ is a metric space with the metric

   $$d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2),$$

   where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

4. **Normalizing a Metric Space**:

   (a) Recall that a continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ is concave if for all $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$

   $$(1-\alpha)f(x) + \alpha f(y) \leq f\left((1-\alpha)x + \alpha y\right).$$

   What does this inequality mean geometrically?

   (b) Prove that if $f(0) \geq 0$ and f is concave then for all $t \in [0, 1]$:

   $$f(tx) \geq t f(x).$$

   (c) Prove that if $f(0) \geq 0$ and f is concave then for all $a, b \geq 0$:

   $$f(a + b) \leq f(a) + f(b).$$

   (d) Show that if (X, d) is a metric space then (X, ρ) is a metric space, where

   $$\rho(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}.$$

5. **Convergence**: Let x_n be a sequence in $\mathbb{R}$. Recall that the **limit superior** of x_n, or $\limsup x_n$, and **limit inferior** of x_n, or $\liminf$, are defined by

   $$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\},$$
   $$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k : k \geq n\}.$$

   (a) Prove that if x_n is bounded then $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ exist in the sense that they are both finite numbers.

   (b) Prove that

   $$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n.$$

   (c) Let x_n be a sequence of real numbers. A point $c \in \mathbb{R} \bigcup \{\pm\infty\}$ is called a **cluster point** of x_n if there is a convergent subsequence of x_n with limit c. Let C denote the of set of cluster points of x_n. Prove that

   $$\limsup_{n\to\infty} x_n = \max C \text{ and } \liminf_{n\to\infty} x_n = \min C.$$

   (d) If x_n is a sequence of real numbers such that

   $$\lim_{n\to\infty} x_n = x,$$

   and $a_n \leq x_n \leq b_n$, prove that

   $$\limsup_{n\to\infty} a_n \leq x \leq \liminf_{n\to\infty} b_n.$$