Homework 2

Analysis

Due: January 29, 2018

1. Metric Spaces:
   (a) Prove that \((\mathbb{R}, d)\) is a metric space with metric \(d(x, y) = |x - y|\).
   (b) Prove that \((\mathbb{R}^n, d_1)\) is a metric space with metric
   \[ d_1(x, y) = \sum_{i=1}^{n} |y_i - x_i|. \]
   (c) Prove that \((\mathbb{R}^n, d_{\infty})\) is a metric space with metric
   \[ d_{\infty}(x, y) = \max_{i \in \{1, \ldots, n\}} |y_i - x_i|. \]
   (d) Let \(X\) be the set of \(n\)-letter words in a \(k\) character alphabet \(A = \{a_1, a_2, \ldots, a_k\}\), meaning that
   \(X = \{(x_1, x_2, \ldots, x_n) : x_i \in A\}\). We define the distance between two words \(x = (x_1, \ldots, x_n)\) and
   \(y = (y_1, \ldots, y_n)\) to be the number of places in which the words have different letters. That is,
   \[ d(x, y) = \#\{i : x_i \neq y_i\}. \]
   Prove that \((X, d)\) is a metric space.

2. Discrete Metric:
   (a) Prove that \((\mathbb{R}^n, d)\) is a metric space with metric
   \[ d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}. \]
   This metric space is called the discrete metric.
   (b) Show that in this metric space topology every subset of \(\mathbb{R}^n\) is open.
   (c) Show that in this metric space topology a sequence \(x_n\) converges to \(x \in \mathbb{R}^n\) if and only if there exists \(N \in \mathbb{N}\) such that \(n \geq N\) implies \(x_n = x\).

3. Building Metric Spaces: \((X, d_X)\) and \((Y, d_Y)\) are metric spaces.
   (a) If \(U \subseteq X\), show that \((U, d_U)\) is a metric space where \(d_U(u_1, u_2) = d_X(u_1, u_2)\) for all \(u_1, u_2 \in U\).
   (b) Show that the Cartesian product \(X \times Y\) is a metric space with the metric
   \[ d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2), \]
   where \(z_1 = (x_1, y_1)\) and \(z_2 = (x_2, y_2)\).
4. Normalizing a Metric Space:

(a) Recall that a continuous function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is concave if for all \( x, y \in \mathbb{R} \) and \( \alpha \in [0, 1] \)

\[
(1 - \alpha)f(x) + \alpha f(y) \leq f((1 - \alpha)x + \alpha y).
\]

What does this inequality mean geometrically?

(b) Prove that if \( f(0) \geq 0 \) and \( f \) is concave then for all \( t \in [0, 1] \):

\[
f(tx) \geq tf(x).
\]

(c) Prove that if \( f(0) \geq 0 \) and \( f \) is concave then for all \( a, b \geq 0 \):

\[
f(a + b) \leq f(a) + f(b).
\]

(d) Show that if \((X, d)\) is a metric space then \((X, \rho)\) is a metric space, where

\[
\rho(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}.
\]

5. Convergence: Let \( x_n \) be a sequence in \( \mathbb{R} \). Recall that the limit superior of \( x_n \), or \( \limsup x_n \), and limit inferior of \( x_n \), or \( \liminf \), are defined by

\[
\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup \{x_k : k \geq n\},
\]

\[
\liminf_{n \to \infty} x_n = \lim_{n \to \infty} \inf \{x_k : k \geq n\}.
\]

(a) Prove that if \( x_n \) is bounded then \( \limsup_{n \to \infty} x_n \) and \( \liminf_{n \to \infty} x_n \) exist in the sense that they are both finite numbers.

(b) Prove that

\[
\liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n.
\]

(c) Let \( x_n \) be a sequence of real numbers. A point \( c \in \mathbb{R} \cup \{\pm \infty\} \) is called a cluster point of \( x_n \) if there is a convergent subsequence of \( x_n \) with limit \( c \). Let \( C \) denote the of set of cluster points of \( x_n \). Prove that

\[
\limsup_{n \to \infty} x_n = \max C \quad \text{and} \quad \liminf_{n \to \infty} x_n = \min C.
\]

(d) If \( x_n \) is a sequence of real numbers such that

\[
\lim_{n \to \infty} x_n = x,
\]

and \( a_n \leq x_n \leq b_n \), prove that

\[
\limsup_{n \to \infty} a_n \leq x \leq \liminf_{n \to \infty} b_n.
\]