Homework 3

Analysis

Due: February 05, 2018

1. **Continuous Mappings**: A mapping between metric spaces $f : (X, d_X) \mapsto (Y, d_Y)$ is continuous if $x_n \to x$ in (X, d_X) implies that $f(x_n) \to f(x)$ in (Y, d_Y).
   (a) Propose an $\varepsilon - \delta$ definition for continuity and prove that your definition agrees with the above characterization.
   (b) Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces and let $f : X \mapsto Y$, and $g : Y \mapsto Z$ be continuous functions. Show that the composition
   $$h = g \circ f : X \mapsto Z,$$
   defined by $h(x) = g(f(x))$, is also continuous.
   (c) Let x_0 be a given point in a metric space (X, d). Show that the function $f : X \mapsto \mathbb{R}$ defined by $f(x) = d(x, x_0)$ is a continuous function.

2. **Completeness and Compactness**:
   (a) Let (X, d) be a metric space and suppose $K \subset X$ is compact with respect to this metric. Prove that a closed subset of K is compact.
   (b) Let (X, d) be a complete metric space, and $Y \subset X$. Prove that (Y, d) is complete if and only if Y is a closed subset of X.
   (c) Suppose that x_n is a sequence in a compact metric space with the property that every convergent subsequence has the same limit x. Prove that $x_n \to x$ as $n \to \infty$.
   (d) Let $f : (X, d_X) \mapsto (Y, d_Y)$ be a continuous mapping and assume X is compact. Prove that the range $f(X)$ is a compact subset of Y.
   (e) Let $f : (X, d) \mapsto \mathbb{R}$ be a continuous mapping and assume X is compact. Prove that f is bounded and obtains its maximum and minimum values.

3. **Geometry of Norms**:
   (a) What is the largest r for which the l^2 circle $\{\mathbf{x} \in \mathbb{R}^2 : \|x\|_2 = r\}$ fits into the the l^1 unit ball on $\mathbb{R}^2$. What is the "radius" of the largest l^1 circle $\{\mathbf{x} : \|x\|_1 = r\}$ that will fit into the l^2 unit ball on $\mathbb{R}^2$.
   (b) Prove that
   $$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \left(|x_1 - y_1|^{\frac{1}{3}} + |x_2 - y_2|^{\frac{1}{3}}\right)^3$$
   is not a metric on $\mathbb{R}^2$.
   (c) Take the points $\mathbf{x} = (1, 0)$, $\mathbf{y} = (0, 1)$. Let d be one of the metrics l^p metrics for $p \geq 1$, or let it be the corresponding non-metric for $0 < p < 1$. Show that
   $$\begin{cases} d(\mathbf{x}, \mathbf{y}) < \mathbf{d}(\mathbf{x}, 0) + d(0, \mathbf{y}) & \text{if } p > 1 \\ d(\mathbf{x}, \mathbf{y}) = \mathbf{d}(\mathbf{x}, 0) + d(0, \mathbf{y}) & \text{if } p = 1 \\ d(\mathbf{x}, \mathbf{y}) > \mathbf{d}(\mathbf{x}, 0) + d(0, \mathbf{y}) & \text{if } p < 1 \end{cases}.$$
   Draw the unit balls centered on the origin, on $\mathbf{x}$, and on $\mathbf{y}$ in each of the three cases. Notice how they "bend" the wrong way when $p < 1$.

(d) Find the shortest distance, in the l^1 metric on $\mathbb{R}^2$, from the origin to the line $x_1 + x_2 = 2$. In the l^∞ metric. In the l^2 metric. Is the shortest distance in the l^p metric a monotone function of p?

(e) Let $(X, \|\cdot\|)$ be a normed linear space. A set $C \subset X$ is convex if for all $x, y \in C$ and all real numbers $0 \le t \le 1$:

$$tx + (1-t)y \in C,$$

i.e. the line segment joining any two points in the C lies in C. Prove that unit ball with respect to $\|\cdot\|$ is convex.

4. **Equivalence of Norms**: Let $\mathbf{x} \in \mathbb{R}^n$.

(a) Show that

$$\lim_{p\to\infty} \|\mathbf{x}\|_p = \max\{|x_1|, \dots \|x_n\|\}.$$

(b) Show that, for all $p, q \ge 1$

$$\frac{\|\mathbf{x}\|_p}{n} \le \|\mathbf{x}\|_q \le n\|\mathbf{x}\|_p.$$

(c) Show that for all $p, q \ge 1$, if $\mathbf{x}_n \to \mathbf{x}$ with respect to the norm $\|\cdot\|_p$ then $\mathbf{x}_n \to \mathbf{x}$ with respect to the norm $\|\cdot\|_q$.