Homework 4
Analysis
Due: February 12, 2018

1. Linearity:
   (a) Let $V$ be the set of continuous real-valued functions on $[0,1]$ satisfying $\int_0^1 x f(x) \, dx = 0$. Is $V$ a linear space?
   (b) Let $V$ be the set of real-valued, twice differentiable functions on $[0,1]$ that are solutions of the differential equation
   \[ y''(x) + xy(x) = 0. \]
   Is $V$ a linear space?
   (c) Let $V$ be the set of real-valued, twice differentiable functions on $[0,1]$ that are solutions of the differential equation
   \[ y''(x) + xy(x) = \sin(x). \]
   Is $V$ a linear space?

2. Completeness and Compactness:
   (a) Show that $\mathbb{R}^n$ is a complete metric space in the $l^p$ metric for $p \geq 1$ (including $p = \infty$).
   (b) Show that $l^1(\mathbb{N}, \mathbb{R})$ is a complete metric space.
   (c) Show that the set \( \{ x \in l^1(\mathbb{N}, \mathbb{R}) : \|x\|_1 \leq 1 \} \), is not compact with respect to the $l^1$ norm.

3. Minimization:
   (a) A function $f : \mathbb{R}^n \to \mathbb{R}$ is coercive if
   \[ \lim_{\|x\| \to \infty} f(x) = \infty. \]
   Explicitly, this condition means that for any $M > 0$ there is an $R > 0$ such that $\|x\| \geq R$ implies $f(x) \geq M$. Prove that if $f$ is lower semicontinuous with respect to this metric and coercive, then $f$ is bounded from below and attains its infimum.
   (b) Let $p : \mathbb{R}^2 \to \mathbb{R}$ be a polynomial function of two real variables. Suppose that $p(x, y) \geq 0$ for all $x, y \in \mathbb{R}$. Does every such function attain its infimum? Prove or disprove.

4. Matrix Norms:
   (a) Let $A \in \mathbb{R}^{n \times n}$. Prove that the following definitions of a matrix norm are identical:
   \[ \|A\| = \max_{x \in \mathbb{R}^n} \frac{\|Ax\|}{\|x\|}, \]
   \[ \|A\| = \max_{\|x\|=1} \|Ax\|, \]
   \[ \|A\| = \max_{\|x\| \leq 1} \frac{\|Ax\|}{\|x\|}. \]
Note: The norm on $\mathbb{R}^n$ does not matter.

(b) Suppose $D \in \mathbb{R}^{n \times n}$ is a diagonal matrix with diagonal entries $d_1, \ldots, d_n$. Prove that for all $p \in [1, \infty]$: 
\[ \|D\|_p = \max_{1 \leq i \leq n} |d_i|. \]

(c) Show that for all $A \in \mathbb{R}^{n \times n}$ that 
\[ \|A\|_{\infty} \leq \sqrt{n} \|A\|_2. \]

(d) For a matrix $A \in \mathbb{R}^{n \times n}$ with entries $a_{ij}$ show that
\begin{itemize}
  \item $\|A\|_{\infty} \leq \max_i \sum_{j=1}^n |a_{ij}|.$
  \item $\|A\|_1 \leq \max_j \sum_{i=1}^n |a_{ij}|.$
  \item Show that there always exists $x \in \mathbb{R}^n$ for which $\|A\|_{\infty} = \max_i \sum_{j=1}^n |a_{ij}|$. Deduce that $\|A\|_{\infty} = \max_i \sum_{j=1}^n |a_{ij}|.$
  \item Prove in a similar way $\|A\|_1 = \max_j \sum_{i=1}^n |a_{ij}|.$
\end{itemize}

5. Sequence spaces: $c_0 \subset \mathbb{R}^n$ is the set of real valued sequences $\{x_n\}$ such that $\lim_{n \to \infty} x_n = 0$.

(a) If $x, y \in c_0$, show that $x + y \in c_0$.

(b) Show that $l^p(\mathbb{N}, \mathbb{R}) \subset c_0$ for all $1 \leq p < \infty$ and that the containment is strict.

(c) Show the $c_0$ is complete in the $l^\infty$ norm.