Homework 4

Analysis

Due: February 12, 2018

1. **Linearity:**
   (a) Let V be the set of continuous real-valued functions on $[0,1]$ satisfying $\int_0^1 xf(x)\,dx = 0$. Is V a linear space?
   (b) Let V be the set of real-valued, twice differentiable functions on $[0,1]$ that are solutions of the differential equation
   $$y''(x) + xy(x) = 0.$$
   Is V a linear space?
   (c) Let V be the set of real-valued, twice differentiable functions on $[0,1]$ that are solutions of the differential equation
   $$y''(x) + xy(x) = \sin(x).$$
   Is V a linear space?
2. **Completeness and Compactness**:
   (a) Show that $\mathbb{R}^n$ is a complete metric space in the l^p metric for $p \geq 1$ (including $p = \infty$).
   (b) Show that $l^1(\mathbb{N}, \mathbb{R})$ is a complete metric space.
   (c) Show that the set $\{x \in l^1(\mathbb{N}, \mathbb{R}) : \|x\|_1 \leq 1)$, is not compact with respect to the l^1 norm.
3. **Minimization**:
   (a) A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is coercive if
   $$\lim_{\|x\|\to\infty} f(x) = \infty.$$
   Explicitly, this condition means that for any $M > 0$ there is an $R > 0$ such that $\|x\| \geq R$ implies $f(x) \geq M$. Prove that if f is lower semicontinuous with respect to this metric and coercive, then f is bounded from below and attains its infimum.
   (b) Let $p : \mathbb{R}^2 \mapsto \mathbb{R}$ be a polynomial function of two real variables. Suppose that $p(x, y) \geq 0$ for all $x, y \in \mathbb{R}$. Does every such function attain its infimum? Prove or disprove.
4. **Matrix Norms**:
   (a) Let $A \in \mathbb{R}^{n\times n}$. Prove that the following definitions of a matrix norm are identical:
   - $\|A\| = \max\limits_{\mathbf{x}\in\mathbb{R}^n} \dfrac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$.
   - $\|A\| = \max\limits_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$.
   - $\|A\| = \max\limits_{\|\mathbf{x}\|\leq 1} \dfrac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$.

Note: The norm on $\mathbb{R}^n$ does not matter.

(b) Suppose $D \in \mathbb{R}^{n\times n}$ is a diagonal matrix with diagonal entries $d_1, \ldots, d_n$. Prove that for all $p \in [1, \infty]$:

$$\|D\|_p = \max_{1\leq i\leq n} |d_i|.$$

(c) Show that for all $A \in \mathbb{R}^{n\times n}$ that

$$\|A\|_\infty \leq \sqrt{n}\|A\|_2.$$

(d) For a matrix $A \in \mathbb{R}^{n\times n}$ with entries a_{ij} show that

- $\|A\|_\infty \leq \max_i \sum_{j=1}^{n} |a_{ij}|$.
- $\|A\|_1 \leq \max_j \sum_{i=1}^{n} |a_{ij}|$.
- Show that there always exists $x \in \mathbb{R}^n$ for which $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$. Deduce that $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$.
- Prove in a similar way $\|A\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$.

5. **Sequence spaces**: $c_0 \subset \mathbb{R}^{\mathbb{N}}$ is the set of real valued sequences $\{x_n\}$ such that $\lim_{n\to\infty} x_n = 0$.

(a) If $x, y \in c_0$, show that $x + y \in c_0$.

(b) Show that $l^p(\mathbb{N}, \mathbb{R}) \subset c_0$ for all $1 \leq p < \infty$ and that the containment is strict.

(c) Show the c_0 is complete in the l^∞ norm.