Homework 5

Analysis

Due: February 23, 2018

1. **Sequence Spaces:**
   (a) Given $1 \leq p < q < \infty$ give an example of a sequence x_n which is in l^q but not in l^p.
   (b) Given $1 \leq p < q < \infty$ prove that $l^p \subset l^\infty$.
   (c) Given $1 \leq p < q < \infty$ prove that $l^p \subset l^q$.
   (d) Let $\mathbf{x}$ be a real valued sequence with components x_n. Suppose for q satisfying $1 \leq q < \infty$ we know that for every $\mathbf{y} \in l^q$ with components y_i the sequence $x_i y_i$ is absolutely summable, and
   $$\left| \sum_{i=1}^{\infty} x_i y_i \right| \leq C \|y\|_{l^q}$$
   for some constant C. Show that $x \in l^p$ where $p = q/(q-1)$.

2. **Young's Inequality**:
   (a) Let f be a continuous strictly increasing function on $[0, \infty)$ with $f(0) = 0$. Prove that for $a, b > 0$:
   $$ab \leq \int_0^a f(t)\,dt + \int_0^b f^{-1}(t)\,dt.$$
   (b) Prove that for all $a, b \geq 0$:
   $$\exp(a) + (1+b)\ln(1+b) \geq (1+a)(1+b).$$

3. **Hölder's and Minkowski's inequalities on spaces of functions**:
   (a) If $f, g \in C([0,1])$, show that
   $$\int_0^1 |f(t)g(t)|\,dt \leq \|f\|_{L^p} \|g\|_{L^q}$$
   with $1/p + 1/q = 1$, $1 < p < \infty$, where
   $$\|h\|_{L^m} = \left(\int_0^1 |h(t)|^m\,dt \right)^{1/m}.$$
   (b) If $f, g \in C([0,1])$ and $1 < p < \infty$, prove that
   $$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

4. **Function Spaces**:
   (a) Let $f_n \in C([a,b])$ be a sequence of functions converging uniformly to a function f. Show that
   $$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

(b) Consider the space of continuously differentiable functions:

$$C^1([a,b]) = \{f : [a,b] \mapsto \mathbb{R} : f, f' \in C([a,b])\}.$$

with the C^1 norm:

$$\|f\|_{C^1} = \sup_{a\le x\le b} |f(x)| + \sup_{a\le x\le b} |f'(x)|.$$

Prove that $C^1([a,b])$ is a Banach space.

(c) Show that the space $C([a,b])$ with the L^1 norm $\|\cdot\|_{L^1}$ defined by

$$\|f\|_{L^1} = \int_a^b |f(x)|\,dx,$$

is incomplete. Show that if $f_n \to f$ with respect to $\|\cdot\|_{L^\infty}$, then $f_n \to f$ with respect to $\|\cdot\|_{L^1}$.

5. **Lebesgue Spaces:**

(a) For each of the following functions f, find the set of $p \in [1,\infty]$ for which $f \in L^p_0$ on the given interval.

- $f(t) = e^{-t}$ on $[0,\infty)$.
- $f(t) = 1/t$ on $[1,\infty)$.
- $f(t) = 1/\sqrt{t}$ on $(0,1]$.
- $f(t) = \ln(t)$ on $(0,3]$.
- $f(t) = 1/\sqrt{t}$ on $(0,\infty)$.
- $f(t) = \begin{cases} |t|^{-1/2} & |t| \le 1 \\ 2t^{-2} & |t| > 1 \end{cases}$ on $(0,\infty)$.

(b) For each of the following functions f, find the set of $\alpha \in \mathbb{R}$ for which $f \in L^p_0$ on the given interval, taking i) $p = 1$, ii) $p = 2$, iii) $p = \infty$.

- $f(t) = t^\alpha$ on $(0,1]$.
- $f(t) = t^\alpha$ on $[1,\infty)$.
- $f(t) = t^\alpha$ on $(0,\infty)$.
- $f(t) = \begin{cases} t^\alpha & 0 < t \le 1 \\ t^{-\alpha} & 1 < t < \infty \end{cases}$ on $[1,\infty)$.

(c) Is there a function f on $(0,\infty)$ which belongs to L^1_0 but not to L^2_0. Is there one that belongs to L^2_0 but not to L^1_0?

(d) Is there a function f on $(0,1)$ which belongs to L^1_0 but not to L^2_0. Is there one that belongs to L^2_0 but not to L^1_0?