Homework 7

Analysis

Due: March 19, 2018

1. **Displacement of a String:** A string of length L is clamped at its endpoints. It is under tension $T > 0$ and is subject to a transverse force per unit length $\tau(x) \geq 0$. The equilibrium displacement $u(x)$ then satisfies the following boundary value problem

$$\begin{aligned} Tu''(x) &= -\tau(x), \\ u(0) &= u(L) = 0. \end{aligned}$$

   Define the energy norm for this problem by:

$$\|f\|_E = \sqrt{\int_0^L [f'(x)]^2 \, dx}.$$

   (a) Assuming there exists a smooth solution $u(x)$ to this problem, prove that

$$\|u\|_E = \sqrt{\frac{1}{T}\int_0^L \tau(x)u(x)\, dx}.$$

   (b) Assuming there exists a smooth solution $u(x)$ to this problem, find upper bounds for the maximum displacement and also the total energy $\mathcal{E}[u] = T\|u\|_E^2/2$ in terms of the tension T, the length L of the string, and the root-mean applied forcing:

$$\overline{\tau} = \sqrt{\frac{1}{L}\int_0^L [\tau(x)]^2\, dx}.$$

2. **Heat Equation:** Assume that the heat equation, and the associate boundary conditions

$$\begin{aligned} \frac{1}{2}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial t}, \\ u(x,0) &= u_0(x) \\ u(0,t) &= u(L,t) = 0 \end{aligned}$$

   has a smooth solution $u(x,t)$ for all smooth initial conditions $u_0(x)$.

   (a) Show that the "spatial" L^2 and energy norms, decay as a function of time, and use this to prove the uniqueness of solutions of the heat equation.

   (b) Show that there is a constant C such that

$$\|u(\cdot,t)\|_{L^2} \leq \|u_0\|_{L^2} e^{-Ct}.$$

   (c) Show that the map $S_t : u_0 \mapsto u(x,t)$ is a continuous map from $L^2([0,L])$ into itself for all $t \geq 0$.

3. **Wave Equation:** Let $b > 0$. The wave equation with dissipation is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t},$$

with the initial boundary conditions:

$$\begin{aligned} u(x,0) &= u_0(x) \\ u_t(x,0) &= v_0(x) \\ u(0,t) &= u(L,t) = 0. \end{aligned}$$

Show the uniqueness of smooth solutions of this equation by constructing your own appropriate energy norm.

4. **Weierstrass Approximation Theorem:** Let

$$p_n(x) = \begin{cases} \left(1 - \frac{x^2}{4}\right)^n, & -2 \leq x \leq 2 \\ 0, & x < -2 \text{ or } x > 2 \end{cases},$$

$c_n = \int_{-\infty}^{\infty} p_n(x)\,dx$ and define $g_n(x)$ by $g_n(x) = \frac{1}{c_n} p_n(x)$.

(a) Let f be a continuous function on $\mathbb{R}$ compactly supported on $[-2, 2]$. Prove that $f_n = g_n * f$ is a polynomial on $[-1, 1]$.

(b) Prove that g_n is a sequence of averaging kernels.

(c) Prove that polynomials are dense in $C([-1, 1])$.

(d) Prove that $C([-1, 1])$ is separable, i.e. it is a dense countable subset.