Homework 8
Analysis
Due: March 26, 2018

1. Convolutions:

(a) Let \( f, g \) be smooth functions with compact support. Let \( A \) be the closure of the set
\[
\{ x + y : x \in \text{supp}(f) \text{ and } y \in \text{supp}(g) \}
\]
Prove that \( \text{supp}(f * g) \subset A \).

(b) Draw a picture of a smooth function \( f \) on \( \mathbb{R} \) satisfying
- \( f \) has compact support.
- For all \( x \in \mathbb{R}, \ 0 \leq f(x) \leq 1 \).

Draw a picture of \( f * f \).

(c) Let \( f = \chi_{[-1,1]} \). Find \( f * f \) without calculating anything. I.e, try to just draw \( f * f \) to obtain the formula for \( f * f \).

(d) Let \( f, g \in C(\mathbb{R}) \) be smooth functions with compact support. Prove that
\[
\| f * g \|_{\infty} \leq \| f \|_{L^p} \| g \|_{L^q},
\]
where \( \frac{1}{p} + \frac{1}{q} = 1 \).

2. Equivalence Classes:

(a) In a metric space \((M, d)\), say that \( x \sim y \) if \( d(x, y) < 1 \). Is this an equivalence relation?

(b) Let \( X \) be the set of \( 2 \times 2 \) complex valued matrices. Say that \( A \sim B \) if \( B = CAC^{-1} \) for some invertible matrix \( C \). Prove that \( \sim \) is an equivalence relation. Prove that the function \( f(A) = \text{trace}(A) \) is defined unambiguously on the set of equivalence classes as well.

3. Abstract Completions:

(a) Recall, a metric space \((X, d)\) is called bounded if there is a \( K > 0 \) such that \( d(x, y) \leq K \) for all \( x, y \in X \). Let \( (X, d) \) be bounded and suppose that \((X, d)\) and \((X', d')\) are isometric. Show that \((X', d')\) is bounded.

(b) Let \((X, d)\) be metric space with completion \((\tilde{X}, \tilde{d})\). Suppose that \((X', d')\) is a complete metric space, and suppose that there is an isometry \( F : X \mapsto X' \) whose range \( F(X) \) is dense in \( X' \). Prove that \((X', d')\) and \((\tilde{X}, \tilde{d})\) are isometric.

(c) Prove that
\[
\begin{align*}
    d(x, y) &= \frac{|x - y|}{\sqrt{(1 + x^2)(1 + y^2)}}
\end{align*}
\]
defines a metric on \( \mathbb{R} \). Show that \( \mathbb{R} \) is not complete in this metric. Find the completion.