Homework 8

Analysis

Due: March 26, 2018

1. **Convolutions:**

   (a) Let f, g be smooth functions with compact support. Let A be the closure of the set

   $$\{x + y : x \in \text{supp}(f) \text{ and } y \in \text{supp}(g)\}$$

   Prove that $\text{supp}(f * g) \subset A$.

   (b) Draw a picture of a smooth function f on $\mathbb{R}$ satisfying

   - f has compact support.
   - For all $x \in \mathbb{R}$, $0 \leq f(x) \leq 1$.

   Draw a picture of $f * f$.

   (c) Let $f = \chi_{[-1,1]}$. Find $f * f$ without calculating anything. I.e, try to just draw $f * f$ to obtain the formula for $f * f$.

   (d) Let $f, g \in C(\mathbb{R})$ be smooth functions with compact support. Prove that

   $$\|f * g\|_\infty \leq \|f\|_{L^p} \|g\|_{L^q},$$

   where $\frac{1}{p} + \frac{1}{q} = 1$.

2. **Equivalence Classes:**

   (a) In a metric space (M, d), say that $x \sim y$ if $d(x, y) < 1$. Is this an equivalence relation?

   (b) Let X be the set of 2×2 complex valued matrices. Say that $A \sim B$ if $B = CAC^{-1}$ for some invertible matrix C. Prove that $\sim$ is an equivalence relation. Prove that the function $f(A) = \text{trace}(A)$ is defined unambiguously on the set of equivalence classes as well.

3. **Abstract Completions:**

   (a) Recall, a metric space (X, d) is called bounded if there is a $K > 0$ such that $d(x, y) \leq K$ for all $x, y \in X$. Let (X, d) be bounded and suppose that (X, d) and (X', d') are isometric. Show that (X', d') is bounded.

   (b) Let (X, d) be metric space with completion $(\tilde{X}, \tilde{d})$. Suppose that (X', d') is a complete metric space, and suppose that there is an isometry $F : X \mapsto X'$ whose range $F(X)$ is dense in X'. Prove that (X', d') and $(\tilde{X}, \tilde{d})$ are isometric.

   (c) Prove that

   $$d(x, y) = \frac{|x - y|}{\sqrt{(1 + x^2)(1 + y^2)}}$$

   defines a metric on $\mathbb{R}$. Show that $\mathbb{R}$ is not complete in this metric. Find the completion.