1. If $T$ is a contraction mapping for $c < 1$, show that we can estimate the distance of the fixed point $x^*$ from the initial point $x_0$ of the fixed point iteration by

$$d(x^*, x_0) \leq \frac{1}{1-c} d(x_1, x_0).$$

2. Using the contraction mapping theorem, show that the following differential equation has a unique solution

$$\begin{cases} \frac{dx}{dt} = \frac{1}{4} (1 + \sin^2(x)) \\ x(0) = 1 \end{cases}.$$

3. The following integral equation for $f : [-a, a] \mapsto \mathbb{R}$ arises in a model of the motion of gas particles on a line:

$$f(x) = 1 + \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x - y)^2} f(y) dy,$$

for $-a \leq x \leq a$. Prove that this equation has a unique bounded, continuous solution for every $0 < a < \infty$. Prove that the solution is nonnegative. What can you say if $a = \infty$?

4. Logistic Equation: The logistic map $T : [0, 1] \mapsto [0, 1]$ is defined by

$$x_{n+1} = T x_n = r x_n (1 - x_n),$$

where $r > 0$.

(a) Show that for $0 < r < 1$ there is a single fixed point $x^*$ for this problem. Prove that if $0 < r < 1$ then for all $x_0 \in [0, 1]$

$$\lim_{n \to \infty} T^n x_0 = x^*.$$

(b) For what values of $r$ will there be exactly two fixed points for this problem. For what range of $r$ will at least one of these fixed points be stable.

(c) A period two orbit is a point $x^*$ satisfying $T^2 x^* = x^*$. For what ranges of $r$ will there exist period two orbits. For what ranges of $r$ will there exist a stable period two orbit?