Homework 9

Analysis

Due: April 09, 2018

1. If T is a contraction mapping for $c < 1$, show that we can estimate the distance of the fixed point x^* from the initial point x_0 of the fixed point iteration by

$$d(x^*, x_0) \leq \frac{1}{1-c} d(x_1, x_0).$$

2. Using the contraction mapping theorem, show that the following differential equation has a unique solution

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{1}{4}\left(1 + \sin^2(x)\right) \\ x(0) = 1 \end{cases}.$$

3. The following integral equation for $f : [-a, a] \mapsto \mathbb{R}$ arises in a model of the motion of gas particles on a line:

$$f(x) = 1 + \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x-y)^2} f(y)\, dy,$$

for $-a \leq x \leq a$. Prove that this equation has a unique bounded, continuous solution for every $0 < a < \infty$. Prove that the solution is nonnegative. What can you say if $a = \infty$?

4. **Logistic Equation:** The logistic map $T : [0,1] \mapsto [0,1]$ is defined by

$$x_{n+1} = Tx_n = rx_n(1 - x_n),$$

where $r > 0$.

   (a) Show that for $0 < r < 1$ there is a single fixed point x^* for this problem. Prove that if $0 < r < 1$ then for all $x_0 \in [0,1]$

$$\lim_{n \to \infty} T^n x_0 = x^*.$$

   (b) For what values of r will there be exactly two fixed points for this problem. For what range of r will at least one of these fixed points be stable.

   (c) A period two orbit is a point x^* satisfying $T^2 x^* = x^*$. For what ranges of r will there exist period two orbits. For what ranges of r will there exist a stable period two orbit?