#1. Let

Draft #1.

$$S_k(x) = \sum_{n=0}^{k} x^n = \frac{1-x^{k+1}}{1-x}.$$

geometric series.

Therefore,

$$\left| S_k(x) - \frac{1}{1-x} \right| = \left| \frac{-x^{k+1}}{1-x} \right| = \left| \frac{x^{k+1}}{1-x} \right| < \varepsilon$$

$$-\varepsilon(1-x) < |x^{k+1}| < \varepsilon(1-x)$$

$$-\varepsilon(1-x)$$

$$|x|^{k+1} < \varepsilon(1-x)$$

$$(k+1)\ln(|x|) < \ln(\varepsilon(1-x)) - 1$$

$$k > \frac{\varepsilon(1-x)}{\ln(|x|)} - 1$$

Scrapwork for finding relationship between k and ε.

possible division by zero.

For all $\varepsilon > 0$, let $M(\varepsilon, x) = \max\left\{ \frac{\ln(\varepsilon(1-x))}{\ln(|x|)} - 1, 0 \right\}$. If $k > M(\varepsilon, x)$ it follows that

max might not be necessary.

$$\left| S_k(x) - \frac{1}{1-x} \right| = \left| \frac{-x^{k+1}}{1-x} \right|$$

$$= \left| \frac{x^{k+1}}{1-x} \right|$$

$$k > \frac{\ln(\varepsilon(1-x))}{\ln(|x|)}$$

$$\Rightarrow \ln(|x|) \cdot k < \ln(\varepsilon(1-x))$$

$$\Rightarrow \ln(|x|^k) < \ln(\varepsilon(1-x))$$

$$\Rightarrow |x|^k < \varepsilon(1-x)$$

$$\Rightarrow \frac{|x|^k}{1-x} < \varepsilon.$$

#2.

$$|x-y| = |x-z+z-y|$$

$$\begin{aligned}
\Rightarrow |x-y|^2 &= (x-z)^2 + 2(x-z)(z-y) + (z-y)^2 \\
&= (x-z)^2 - 2(x-z)(y-z) + (y-z)^2 \\
&\geq (x-z)^2 - 2|x-z|\cdot|y-z| + (y-z)^2 \\
&= (|x-z| - |y-z|)^2
\end{aligned}$$

Therefore,

$$|x-y| \geq \big|\,|x-z| - |y-z|\,\big|$$

↓ used the fact that $\sqrt{\ }$ is an increasing function and $\sqrt{a^2} = |a|$.

#4.

$$\begin{aligned}
|a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} - a_n| \\
&\leq |a_m - a_{m-1}| + |a_{m-1} - a_n| \\
&\leq \frac{1}{(m-1)^2} + |a_{m-1} + a_{m-2} - a_{m-2} - a_n| \\
&\leq \frac{1}{(m-1)^2} + |a_{m-1} - a_{m-2}| + |a_{m-2} - a_n| \\
&\leq \frac{1}{(m-1)^2} + \frac{1}{(m-2)^2} + |a_{m-2} - a_n| \\
&\ \ \vdots \\
&\leq \frac{1}{(m-1)^2} + \frac{1}{(m-2)^2} + \cdots + \frac{1}{n^2} \\
&= \sum_{k=n}^{m-1} \frac{1}{k^2}. \\
&\leq \sum_{k=n}^{\infty} \frac{1}{k^2}.
\end{aligned}$$

Since $\sum_{k=n}^{\infty} \frac{1}{k^2}$ converges it follows that

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{k^2} = 0.$$

which implies a_n is Cauchy.

#5.

$$|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - \dots + a_{n+1} - a_n|$$
$$\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \dots + |a_{n+1} - a_n|$$
$$\leq CK^{m-1} + \dots + CK^n$$
$$\leq C \sum_{i=n}^{m-1} K^i$$
$$\leq C \sum_{i=n}^{\infty} K^i$$

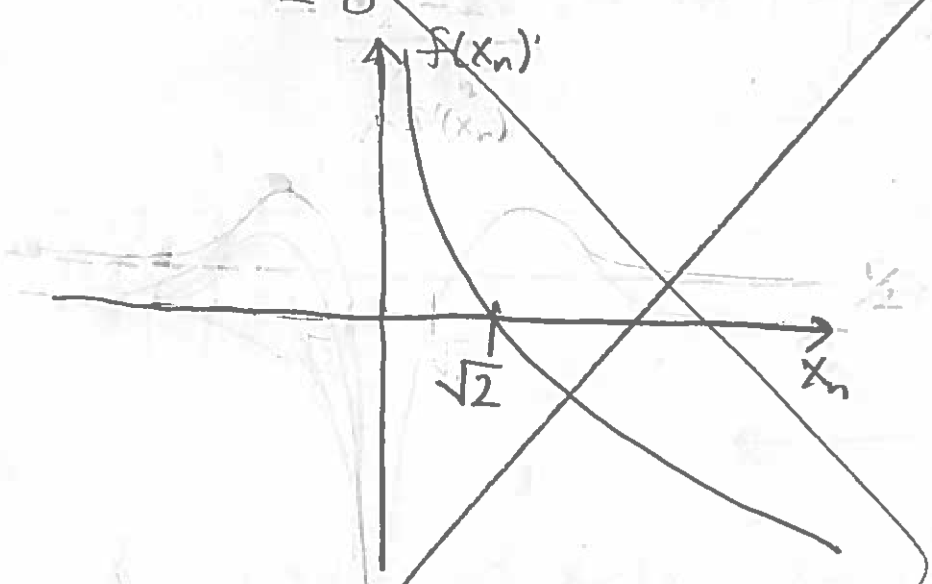
Since $0<K<1$ it follows that $\sum_{i=n}^{\infty} K^i < \infty$ and hence $\lim_{n\to\infty} \sum_{i=n}^{\infty} K^i = 0$.

#6.

Method #1 → Not useful.

$$x_{n+1} - x_n = \frac{x_n}{2} + \frac{1}{x_n} - x_n = f(x_n) = -\frac{x_n}{2} + \frac{1}{x_n} = \frac{1-x_n^2}{2x_n}$$

$$f'(x_n) = -\frac{1}{2} - \frac{1}{x_n^2} = -\frac{(x_n^2+1)}{2x_n^2} < 0$$



This tells me $x_{n+1} - x_n$ is decrasing. However, we need to know how fast it is going to 0.

Now, $f'($

Method #2:

Let $f(x) = \frac{x}{2} + \frac{1}{x}$. Then,

$$x_{n+1} - x_n = f(x_n) - f(x_{n-1})$$

$$= \frac{x_n}{2} + \frac{1}{x_n} - \frac{x_{n-1}}{2} - \frac{1}{x_{n-1}}$$

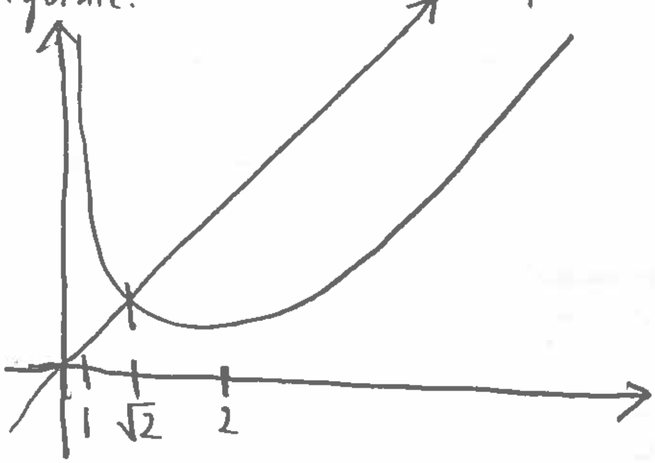
$$= \frac{x_n - x_{n-1}}{2} + \frac{1}{x_n} - \frac{1}{x_{n-1}}$$

$$= \frac{x_n - x_{n-1}}{2} + \frac{x_{n-1} - x_n}{x_n x_{n-1}}$$

$$= \left(\frac{1}{2} - \frac{1}{x_n x_{n-1}}\right) \cdot (x_n - x_{n-1})$$

$$= \frac{1}{2}\left(1 - \frac{2}{x_n x_{n-1}}\right) \cdot (x_n - x_{n-1})$$

I need to show $\left|1 - \frac{2}{x_n x_{n-1}}\right| < 1$ for the values of x_n in the sequence.



Since $f(x) < x$ for $\sqrt{2} \leq x < 2$ and $f(x) > x$ for $1 \leq x \leq \sqrt{2}$ and $f > 0$ it follows that for all $1 \leq x_n \leq 2$ that $1 \leq x_{n+1} \leq 2$

The max and minimum values of $f(x)$ on this interval are given by $3/2$ and $2/\sqrt{2}$.

$$\Rightarrow 2 \leq x_n \cdot x_{n-1} \leq 9/4$$

$$\Rightarrow \frac{8}{9} \leq \frac{2}{x_n x_{n-1}} \leq 1$$

$$\Rightarrow 0 \leq \left|1 - \frac{2}{x_n x_{n-1}}\right| \leq \frac{1}{9}.$$

This will contract so we can finish the proof.