#1.
Give an ε-δ proof that for all \( x \in (-1, 1) \):

\[
\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}
\]

**Proof:**

Let \( S_K(x) = \sum_{n=0}^{K} x^n = \frac{1-x^{K+1}}{1-x} \). Therefore,

\[
\left| S_K(x) - \frac{1}{1-x} \right| = \left| \frac{1-x^{K+1}}{1-x} \right| = \frac{1-x^{K+1}}{1-x} < \frac{1-x}{1-x} = 1
\]

1. For all \( \varepsilon > 0 \), assume \( x \neq 0 \) and let

\[
M(\varepsilon, x) = m(\varepsilon (1-x)) - 1
\]

where \( m(x) = \ln (1+x) \).

If \( K > M(\varepsilon, x) \) it follows that

\[
K > m(\varepsilon (1-x)) - 1 = \ln (1-x) - 1
\]

\[
\Rightarrow (K+1) \ln (1-x) < \ln (\varepsilon (1-x)) \cdot \ln (1+x) \leq 1
\]

\[
\Rightarrow \ln (1-x^{K+1}) < \ln (\varepsilon (1-x))
\]

\[
\Rightarrow 1-x^{K+1} < \varepsilon (1-x)
\]

\[
\Rightarrow \frac{x^{K+1}}{1-x} < \varepsilon
\]

\[
\Rightarrow \frac{x^{K+1}}{1-x} < \varepsilon
\]

2. If \( x=0 \) then \( S_K(x) = 1 \) for all \( K \).

By items 1 and 2 it follows that for all \( x \in (-1,1) \):

\[
\lim_{K \to \infty} \sum_{n=0}^{K} x^n = \frac{1}{1-x}
\]
#2.

Prove for all $x, y, z \in \mathbb{R}$,
\[ |x - y| \geq |x - z| - |z - y| \]

**Proof:**

Let $x, y, z \in \mathbb{R}$. Since $|x - y| = |x - z + z - y|$ it follows that
\[ |x - y|^2 = (x - z + z - y)^2 \]
\[ = (x - z)^2 + 2(x - z)(z - y) + (z - y)^2 \]
\[ \geq (x - z)^2 - 2|x - z| |y - z| + (z - y)^2 \]
\[ = (|x - z| - |z - y|)^2. \]

Therefore, since the square root function is increasing and $\sqrt{a^2} = |a|$ it follows that
\[ |x - y| \geq |x - z| - |z - y| . \]

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#4.

Let $a_n$ be a sequence in $\mathbb{R}$ satisfying for all $n \in \mathbb{N}$,
\[ |a_{n+1} - a_n| < \frac{1}{n^2} \]

Is $a_n$ a Cauchy sequence? Prove or give a counterexample.

**Proof:**

By repeated application of the triangle inequality it follows that for all $m, n \in \mathbb{N}$ with $m > n$ that
\[ |a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - a_n| \]
\[ \leq |a_m - a_{m-1}| + |a_{m-1} - a_n| \]
\[ \leq \frac{1}{(m-1)^2} + |a_{m-1} - a_n| \]
\[ = \frac{1}{(m-1)^2} + |a_{m-1} - a_{m-2} + a_{m-2} - a_n| \]
\[ \leq \frac{1}{(m-1)^2} + |a_{m-1} - a_{m-2}| + |a_{m-2} - a_n| \]
\[ \leq \frac{1}{(m-1)^2} + \frac{1}{(m-2)^2} + |a_{m-2} - a_n| \]
\[ \leq \frac{1}{(m-1)^2} + \frac{1}{(m-2)^2} + \ldots + \frac{1}{n^2} . \]
There fore,
\[ |a_n - a_m| \leq \sum_{k=m}^{n-1} \frac{1}{k^2} \leq \sum_{k=m}^{\infty} \frac{1}{k^2}. \]

Since \( \sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2) < \infty \) it follows that \[ \lim_{n \to \infty} \sum_{k=n}^{\infty} \frac{1}{k^2} = 0. \]

#5.
Draft 1 was O.K.

#6.
Define a sequence of real numbers \( x_1 = 2 \) and \[ x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}. \]

Prove that \( x_n \) is Cauchy.

**Proof:**

1. Calculating it follows that
\[
x_{n+1} - x_n = \frac{x_n}{2} + \frac{1}{x_n} - \frac{x_{n-1}}{2} - \frac{1}{x_{n-1}}
\]
\[
= \frac{x_n - x_{n-1}}{2} + \frac{x_{n-1} - x_n}{x_n x_{n-1}}
\]
\[
= \frac{1}{2} \left( x_n - x_{n-1} \right) \cdot \left( 1 - \frac{2}{x_n x_{n-1}} \right)
\]

2. Let \( f(x) = \frac{x}{2} + \frac{1}{x} \). Calculating, \( f'(x) = \frac{1}{2} - \frac{1}{x^2} \) which on the interval \([1, 2]\) has a zero at \( x = \sqrt{2} \). Now,
\[
f(1) = \frac{3}{2}
\]
\[
f(2) = \frac{3}{2}
\]
\[
f(\sqrt{2}) = \sqrt{2}
\]

Which implies \( x_n \in [\sqrt{2}, \frac{3}{2}] \) for all \( n \geq 2 \).

3. From \( f' \geq 0 \) it follows that
\[
2 \leq x_n x_{n-1} \leq \frac{9}{4}
\]
\[
\Rightarrow \theta = 2 - 1 \rightarrow \theta < 1 - \frac{2}{\theta} < \frac{1}{\theta}
\]
There fore,
\[ |a_n - a_{n+1}| \leq \sum_{k=n}^{m-1} \frac{1}{k^2} \leq \sum_{k=n}^{\infty} \frac{1}{k^2}. \]

Since \( \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty \), it follows that \( \lim_{n \to \infty} \sum_{k=n}^{\infty} \frac{1}{k^2} = 0 \).

\#5.
Draft 1 was O.K.

\#6.
Define a sequence of real numbers \( x_1 = 2 \) and
\[ x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}. \]

Prove that \( x_n \) is Cauchy.

**Proof:**
1. Calculating it follows that
   \[ x_{n+1} - x_n = \frac{x_n}{2} + \frac{1}{x_n} - \frac{x_{n-1}}{2} - \frac{1}{x_{n-1}}. \]
   \[ = x_n - x_{n-1} + \frac{x_{n-1} - x_n}{x_n x_{n-1}}. \]
   \[ = \frac{1}{2} (x_n - x_{n-1}) \cdot \left(1 - \frac{2}{x_n x_{n-1}}\right). \]

2. Let \( f(x) = \frac{x}{2} + \frac{1}{x} \). Calculating, \( f'(x) = \frac{1}{2} - \frac{1}{x^2} \) which on the interval \([1, 2]\) has a zero at \( x = \sqrt{2} \). Now,
   \[ f(1) = \frac{3}{2}, \]
   \[ f(2) = \frac{5}{2}, \]
   \[ f(\sqrt{2}) = \sqrt{2}. \]

Which implies \( x_n \in [\sqrt{2}, \frac{3}{2}] \) for all \( n \geq 2 \).

3. From iter 2 it follows that
   \[ 2 \leq x_n x_{n-1} \leq \frac{9}{4} \]
   \[ \Rightarrow x_n \to 1 - \frac{2}{x_{n-1}} \to \n < 1 - \frac{2}{\sqrt{2}} < \frac{1}{n}. \]
4. By items 1 and 3 it follows that

\[ |x_{n+1} - x_n| = \frac{1}{2} \left| 1 - \frac{2}{x_n x_{n-1}} \right| |x_n - x_{n-1}| \]

\[ \leq \frac{1}{18} |x_n - x_{n-1}|. \]

Therefore, by problem #5, \( x_n \) is a Cauchy sequence.