Homework #1 Draft #2

#1.

Give an ε-δ proof that for all $x \in (-1,1)$:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

proof:

Let $S_K(x) = \sum_{n=0}^{K} x^n = \frac{1-x^{K+1}}{1-x}$ (geometric series). Therefore,

$$\left| S_K(x) - \frac{1}{1-x} \right| = \left| \frac{-x^{K+1}}{1-x} \right| = \left| \frac{x^{K+1}}{1-x} \right| = \frac{|x^{K+1}|}{1-x}$$

$x \in (-1,1)$. $1-x>0$ o.k. to drop absolute values.

1. For all $\varepsilon > 0$, assume $x \neq 0$ and let

$$M(\varepsilon, x) = \frac{\ln(\varepsilon(1-x))}{\ln(|x|)} - 1.$$

If $K > M(\varepsilon, x)$ it follows that

$$K > \frac{\ln(\varepsilon(1-x))}{\ln(|x|)} - 1$$

$$\Rightarrow (K+1)\ln(|x|) < \ln(\varepsilon(1-x))$$ sign change here since $|x|<1$.

$$\Rightarrow \ln(|x|^{K+1}) < \ln(\varepsilon(1-x))$$ $\ln(x)$ is monotone increasing used here.

$$\Rightarrow |x|^{K+1} < \varepsilon(1-x)$$

$$\Rightarrow \frac{|x|^{K+1}}{1-x} < \varepsilon$$

$$\Rightarrow \frac{|x^{K+1}|}{1-x} < \varepsilon$$

2. If $x=0$ then $S_K(x) = 1$ for all K.

By items 1 and 2 it follows that for all $x \in (-1,1)$:

$$\lim_{K\to\infty} \sum_{n=0}^{K} x^n = \frac{1}{1-x}$$

#2.

Prove for all $x, y, z \in \mathbb{R}$,

$$|x-y| \geq ||x-z|-|z-y||.$$

proof:

Let $x,y,z \in \mathbb{R}$. Since $|x-y| = |x-z+z-y|$ it follows that

$$\begin{aligned}
|x-y|^2 &= (x-z+z-y)^2 \\
&= (x-z)^2 + 2(x-z)(z-y) + (z-y)^2 \\
&\geq (x-z)^2 - 2\cdot|x-z|\cdot|y-z| + (z-y)^2 \\
&= (|x-z|-|z-y|)^2.
\end{aligned}$$

Therefore, since the square root function is increasing and $\sqrt{a^2} = |a|$ it follows that

$$|x-y| \geq ||x-z|-|z-y||.$$

#4.

Let a_n be a sequence in $\mathbb{R}$ satisfying for all $n \in \mathbb{N}$:

$$|a_{n+1} - a_n| < \frac{1}{n^2}$$

Is a_n a Cauchy sequence? Prove or give a counterexample.

proof:

By repeated application of the triangle inequality it follows that for all $m,n \in \mathbb{N}$ with $m \geq n$ that

$$\begin{aligned}
|a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} - a_n| \\
&\leq |a_m - a_{m-1}| + |a_{m-1} - a_n| \\
&\leq \frac{1}{(m-1)^2} + |a_{m-1} - a_n| \\
&= \frac{1}{(m-1)^2} + |a_{m-1} - a_{m-2} + a_{m-2} - a_n| \\
&\leq \frac{1}{(m-1)^2} + |a_{m-1} - a_{m-2}| + |a_{m-2} - a_n| \\
&\leq \frac{1}{(m-1)^2} + \frac{1}{(m-2)^2} + |a_{m-2} - a_n| \\
&\vdots \\
&\leq \frac{1}{(m-1)^2} + \frac{1}{(m-2)^2} + \dots + \frac{1}{n^2}.
\end{aligned}$$

There fore,

$$|a_m - a_n| \le \sum_{k=n}^{m-1} \frac{1}{k^2} \le \sum_{k=n}^{\infty} \frac{1}{k^2}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ it follows that $\lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{k^2} = 0.$

#5.

Draft 1 was O.K.

#6.

Define a sequence of real numbers $x_1 = 2$ and

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}.$$

Prove that x_n is Cauchy.

proof:

1. Calculating it follows that

$$\begin{aligned} x_{n+1} - x_n &= \frac{x_n}{2} + \frac{1}{x_n} - \frac{x_{n-1}}{2} - \frac{1}{x_{n-1}} \\ &= \frac{x_n - x_{n-1}}{2} + \frac{x_{n-1} - x_n}{x_n x_{n-1}} \\ &= \frac{1}{2}(x_n - x_{n-1}) \cdot \left(1 - \frac{2}{x_n x_{n-1}}\right) \end{aligned}$$

2. Let $f(x) = \frac{x}{2} + \frac{1}{x}$. Calculating, $f'(x) = \frac{1}{2} - \frac{1}{x^2}$ which on the interval $[1,2]$ has a zero at $x = \sqrt{2}$. Now,

$$f(1) = 3/2$$
$$f(2) = 3/2$$
$$f(\sqrt{2}) = \sqrt{2}$$

Which implies $x_n \in [\sqrt{2}, 3/2]$ for all $n \ge 2$.

3. From item 2 it follows that for

$$2 \le x_n \, x_{n-1} \le 9/4$$

$$\Rightarrow \frac{8}{9} \le \frac{2}{x_n x_{n-1}} \le 1 \Rightarrow 0 \le 1 - \frac{2}{x_n x_{n-1}} \le 1/9$$

There fore,

$$|a_m - a_n| \le \sum_{k=n}^{m-1} \frac{1}{k^2} \le \sum_{k=n}^{\infty} \frac{1}{k^2}.$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$ it follows that $\lim_{n\to\infty} \sum_{k=n}^{\infty} \frac{1}{k^2} = 0.$

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proof:

1. Calculating it follows that

$$\begin{aligned} x_{n+1} - x_n &= \frac{x_n}{2} + \frac{1}{x_n} - \frac{x_{n-1}}{2} - \frac{1}{x_{n-1}} \\ &= \frac{x_n - x_{n-1}}{2} + \frac{x_{n-1} - x_n}{x_n x_{n-1}} \\ &= \frac{1}{2}(x_n - x_{n-1}) \cdot \left(1 - \frac{2}{x_n x_{n-1}}\right) \end{aligned}$$

2. Let $f(x) = \frac{x}{2} + \frac{1}{x}$. Calculating, $f'(x) = \frac{1}{2} - \frac{1}{x^2}$ which on the interval $[1,2]$ has a zero at $x = \sqrt{2}$. Now,

$$f(1) = 3/2$$
$$f(2) = 3/2$$
$$f(\sqrt{2}) = \sqrt{2}$$

Which implies $x_n \in [\sqrt{2}, 3/2]$ for all $n \ge 2$.

3. From item 2 it follows that for

$$2 \le x_n x_{n-1} \le 9/4$$

$\Rightarrow 8 \le 2 \le 1 \Rightarrow 0 \le 1 - 2 \le 1/9$

4. By items 1 and 3 it follows that

$$|X_{n+1} - X_n| = \frac{1}{2} \cdot \left|1 - \frac{2}{X_n X_{n-1}}\right| \cdot |X_n - X_{n-1}|,$$
$$\leq \frac{1}{18} |X_n - X_{n-1}|.$$

Therefore, by problem #5, X_n is a Cauchy sequence.