Homework 5

Analysis

Due: February 23, 2018

1. Sequence Spaces:

(a) Given $1 \leq p < q < \infty$ give an example of a sequence $x_n$ which is in $l^q$ but not in $l^p$.
(b) Given $1 \leq p < q < \infty$ prove that $l^p \subset l^\infty$.
(c) Given $1 \leq p < q < \infty$ prove that $l^p \subset l^q$.
(d) Let $x$ be a real valued sequence with components $x_n$. Suppose for $q$ satisfying $1 \leq q < \infty$ we know that for every $y \in l^q$ with components $y_i$ the sequence $x_i y_i$ is absolutely summable, and

$$\sum_{i=1}^{\infty} x_i y_i \leq C \|y\|_q$$

for some constant $C$. Show that $x \in l^p$ where $p = q/(q-1)$.

2. Young's Inequality:

(a) Let $f$ be a continuous strictly increasing function on $[0, \infty)$ with $f(0) = 0$. Prove that for $a, b > 0$:

$$ab \leq \int_0^a f(t) \, dt + \int_0^b f^{-1}(t) \, dt.$$

(b) Prove that for all $a, b \geq 0$:

$$\exp(a) + (1 + b) \ln(1 + b) \geq (1 + a)(1 + b).$$

3. Hölder's and Minkowski's inequalities on spaces of functions:

(a) If $f, g \in C([0, 1])$, show that

$$\int_0^1 |f(t)g(t)| \, dt \leq \|f\|_{L^p} \|g\|_{L^q},$$

with $1/p + 1/q = 1$, $1 < p < \infty$, where

$$\|h\|_{L^m} = \left( \int_0^1 |h(t)|^m \, dt \right)^{1/m}.$$

(b) If $f, g \in C([0, 1])$ and $1 < p < \infty$, prove that

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

4. Function Spaces:

(a) Let $f_n \in C([a, b])$ be a sequence of functions converging uniformly to a function $f$. Show that

$$\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx.$$
(b) Consider the space of continuously differentiable functions:

\[ C^1([a, b]) = \{ f : [a, b] \to \mathbb{R} : f, f' \in C([a, b]) \} \]

with the \( C^1 \) norm:

\[ \| f \|_{C^1} = \sup_{a \leq x \leq b} |f(x)| + \sup_{a \leq x \leq b} |f'(x)|. \]

Prove that \( C^1([a, b]) \) is a Banach space.

(c) Show that the space \( C([a, b]) \) with the \( L^1 \) norm \( \| \cdot \|_{L^1} \) defined by

\[ \| f \|_{L^1} = \int_a^b |f(x)| \, dx, \]

is incomplete. Show that if \( f_n \to f \) with respect to \( \| \cdot \|_{L^\infty} \), then \( f_n \to f \) with respect to \( \| \cdot \|_{L^1} \).

5. Lebesgue Spaces:

(a) For each of the following functions \( f \), find the set of \( p \in [1, \infty] \) for which \( f \in L^p_0 \) on the given interval.

- \( f(t) = e^{-t} \) on \([0, \infty)\).
- \( f(t) = 1/t \) on \([1, \infty)\).
- \( f(t) = 1/\sqrt{t} \) on \([0, 1]\).
- \( f(t) = \ln(t) \) on \((0, 3]\).
- \( f(t) = 1/\sqrt{t} \) on \((0, \infty)\).
- \( f(t) = \begin{cases} |t|^{-1/2} & |t| \leq 1 \\ 2t^{-2} & |t| > 1 \end{cases} \) on \((0, \infty)\).

(b) For each of the following functions \( f \), find the set of \( \alpha \in \mathbb{R} \) for which \( f \in L^p_0 \) on the given interval, taking i) \( p = 1 \), ii) \( p = 2 \), iii) \( p = \infty \).

- \( f(t) = t^\alpha \) on \((0, 1]\).
- \( f(t) = t^\alpha \) on \([1, \infty)\).
- \( f(t) = t^\alpha \) on \((0, \infty)\).

- \( f(t) = \begin{cases} t^\alpha & 0 < t \leq 1 \\ t^{-\alpha} & 1 < t < \infty \end{cases} \) on \([1, \infty)\).

(c) Is there a function \( f \) on \((0, \infty)\) which belongs to \( L^1_0 \) but not to \( L^2_0 \). Is there one that belongs to \( L^2_0 \) but not to \( L^1_0 \)?

(d) Is there a function \( f \) on \((0, 1)\) which belongs to \( L^1_0 \) but not to \( L^2_0 \). Is there one that belongs to \( L^2_0 \) but not to \( L^1_0 \)?
Homework #5.

#1.

(a) Given $1 \leq p < q < \infty$ give an example of a sequence $x_n$ in $l^q$ but not in $l^p$.

**Solution:**

Let $x_n = \left(\frac{1}{n}\right)^{1/p}$. Therefore, $\|x_n\|_q = \left(\frac{1}{n}\right)^{1/q}$. But $\|x_n\|_p = \frac{1}{n}$.

(b) Given $1 \leq p < q < \infty$ prove that $l^p \subset l^q$.

**Proof:**

If $x_n \in l^p$ then $\lim_{n \to \infty} x_n = 0$ and therefore $\|x_n\|$ is bounded.

(c) Given $1 \leq p < q < \infty$, prove that $l^p \subset l^q$.

**Proof:**

Let $x_n \in l^p$. Therefore,

$$\sum_{n=1}^\infty |x_n|^q = \sum_{n=1}^\infty |x_n|^p |x_n|^{q-p} \leq \sum_{n=1}^\infty |x_n|^p \|x_n\|_\infty^{q-p} = \|x_n\|_\infty^{q-p} \sum_{n=1}^\infty |x_n|^p < \infty.$$  

(d) Let $x_n$ be a real valued sequence. Suppose for $1 \leq q < \infty$ we know that for every $y \in l^q$ the sequence $x_i y_i$ is absolutely summable, and

$$\left| \sum_{i=1}^\infty x_i y_i \right| \leq C \|y\|_{l^q}$$

for some constant $C$. Show that $x \in l^p$ where $p = \frac{q}{q-1}$.

**Proof:**

Let $y^{(n)} \in l^q$ be defined by

$$y^{(n)} = (\text{sgn}(x_1)|x_1|^{p-1}, \text{sgn}(x_2)|x_2|^{p-1}, \ldots, \text{sgn}(x_n)|x_n|^{p-1}, 0, 0, \ldots)$$

Therefore,

$$\left| \sum_{i=1}^\infty x_i y_i \right| = \sum_{i=1}^n |x_i|^p \leq C \left( \sum_{i=1}^n |x_i|^{p-1} \right)^{1/p}$$

$$\Rightarrow \sum_{i=1}^n |x_i|^p \leq C \left( \sum_{i=1}^n |x_i|^p \right)^{1/p}$$

$$\Rightarrow \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \leq C.$$
Therefore, since \((\sum_{i=1}^{n} |x_i|^p)^{1/p}\) is uniformly bounded and monotone increasing in \(n\) it follows that:
\[
\|x\|_p = \lim_{n \to \infty} \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} < \infty.
\]

2. 

a) Let \(f\) be a continuous strictly increasing function on \([0, \infty)\) with \(f(0) = 0\). Prove that for \(a, b > 0\):
\[
ab \leq \int_{a}^{b} f(x)\,dx + \int_{0}^{a} f^{-1}(t)\,dt.
\]

\[\text{Proof:}\]

\[\text{Case 1:}\]
1. \(ab = \int_{0}^{a} f(x)\,dx + \int_{0}^{b} f^{-1}(x)\,dx + a \cdot (b - f(a))\)
\[
\leq \int_{0}^{a} f(x)\,dx + \int_{0}^{b} f^{-1}(x)\,dx
\]
2. \(ab = \int_{0}^{a} f(x)\,dx + \int_{0}^{b} f^{-1}(x)\,dx + b \cdot (a - f^{-1}(b))\)
\[
\leq \int_{0}^{a} f(x)\,dx + \int_{0}^{b} f^{-1}(x)\,dx.
\]

\[\text{Case 2:}\]

b) Prove that for all \(a, b \geq 0\):
\[
\exp(a) + (1 + b) - (1 + b) \geq (1 + a)(1 + b).
\]

\[\text{Proof:}\]

We apply Young's inequality with \(f(x) = e^x - 1\).
\[
\Rightarrow ab \leq \int_{a}^{b} (e^x - 1)\,dx + \int_{0}^{b} x\,(x+1)\,dx
\]
\[
= e^b - e^a - 1 + (1 + b) - (1 + b) - b
\]
\[
\Rightarrow a + ab + b + 1 \leq \exp(a) + (1 + b) - (1 + b)
\]
\[
\Rightarrow (1 + a)(1 + b) \leq \exp(a) + (1 + b) - (1 + b).\]
3.

a) If \( f, g \in C([0,1]) \), show that
\[
\int_0^1 |f(t)g(t)| \, dt \leq ||f||_p ||g||_q
\]
with \( \frac{1}{p} + \frac{1}{q} = 1, \ 1 < p < \infty \).

\[
\frac{1}{||f||_p ||g||_q} \int_0^1 |f(t)g(t)| \, dt = \int_0^1 \left| \frac{f(t)}{||f||_p} \right| \left| \frac{g(t)}{||g||_q} \right| \, dt
\]

Therefore, by Young's inequality:
\[
\left| \frac{f(t)}{||f||_p} \right| \left| \frac{g(t)}{||g||_q} \right| \leq \frac{1}{p} \left| \frac{f(t)}{||f||_p} \right|^p + \frac{1}{q} \left| \frac{g(t)}{||g||_q} \right|^q
\]

\[
\Rightarrow \int_0^1 \left| \frac{f(t)}{||f||_p} \right| \left| \frac{g(t)}{||g||_q} \right| \, dt \leq \frac{1}{p} + \frac{1}{q} = 1
\]

\[
\Rightarrow \int_0^1 |f(t)g(t)| \, dt \leq ||f||_p ||g||_q
\]

b) If \( f, g \in C([0,1]) \) and \( 1 < p < \infty \), show that
\[
||f + g||_p \leq ||f||_p + ||g||_p
\]

Solution:

Choose \( g \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Therefore, by Hölder's inequality:
\[
\int_0^1 |f(t) + g(t)|^p \, dt \leq \left( \int_0^1 |f(t)|^p \, dt \right)^{\frac{1}{p}} \left( \int_0^1 |g(t)|^q \, dt \right)^{\frac{1}{q}}
\]

\[
\Rightarrow \int_0^1 |f(t) + g(t)|^p \, dt \leq ||f||_p + ||g||_p
\]

\[
\Rightarrow ||f + g||_p \leq ||f||_p + ||g||_p
\]
A.) Let \( f_n \in C([a,b]) \) be a sequence of functions converging uniformly to a function \( f \). Show that
\[
\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx.
\]

**Proof:**
\[
\left| \int_a^b f_n(x) \, dx - \int_a^b f(x) \, dx \right| = \left| \int_a^b (f_n(x) - f(x)) \, dx \right|
\leq \int_a^b |f_n(x) - f(x)| \, dx
\leq \int_a^b \|f_n - f\|_{\infty} \, dx
= (b-a) \|f_n - f\|_{\infty}.
\]
The result follows since \( \|f_n - f\|_{\infty} \to 0 \).

b.) Consider the space of continuously differentiable functions:
\[ C^1([a,b]) = \{ f : [a,b] \to \mathbb{R} \mid f, f' \in C([a,b]) \} \]
with the \( C^1 \) norm:
\[ \|f\|_{C^1} = \|f\|_{\infty} + \|f'\|_{\infty}. \]
Prove that \( C^1([a,b]) \) is a Banach space.

**Proof:**
Let \( f_n \) be Cauchy in \( C^1([a,b]) \). Therefore, \( f_n \) and \( f_n' \) are Cauchy in \( C([a,b]) \) and hence converge uniformly to \( f \) and \( g \) respectively. Therefore,
\[ \|f - f_n\|_{C^1} = \|f - f_n\|_{\infty} + \|g - f_n'\|_{\infty} \to 0. \]
To finish the proof we need to show that \( f' = g' \). By the Fundamental Theorem of Calculus:
\[ f_n(t) - f_n(a) = \int_a^t f_n'(x) \, dx. \]
Since \( f_n' \to g' \) uniformly it follows that
\[ f(t) - f(a) = \int_a^t g'(x) \, dx. \]
C.) Show that the space $C([0,1])$ with the $L^1$ norm $\| \cdot \|_1$ is incomplete. Show that if $f_n \to f$ with respect to $\| \cdot \|_\infty$, then $f_n \to f$ with respect to $\| \cdot \|_1$.

**Solution:**

On $[0,1]$ consider the following sequence of functions:

$$f_n(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\ \frac{1}{2} (x - \frac{1}{2} - \frac{1}{n}) & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2} + \frac{1}{n} \\ 1 & \frac{1}{2} + \frac{1}{n} \leq x \leq 1 \end{cases}$$

and define $f$ by:

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} < x \leq 1 \end{cases}$$

Computing, it follows that

$$\int_0^1 |f_n(x) - f(x)| \, dx = \frac{1}{2n}.$$ 

Therefore, $f_n \to f$ in $L^1$, but $f \notin L^1$.

Now, if $f_n \to f$ in $L^\infty$ then,

$$\int_a^b |f_n(x) - f(x)| \, dx \leq \int_a^b \| f_n - f \|_\infty \, dx = (b-a) \| f_n - f \|_\infty$$

$\Rightarrow f_n \to f$ in $L^1$.

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#5.

a.) For each of the following functions $f$, find the set of $p \in [1,\infty]$ for which $f \in L^p$ on the given interval.

i.) $f(x) = e^{-x}$ on $[0,\infty)$.

**Solution:**

$1 \leq p \leq \infty$.

ii.) $f(x) = \frac{1}{x}$ on $[1,\infty)$.

**Solution:**

$1 < p \leq \infty$. 
\[ \text{iii). } f(x) = \frac{1}{x^2} \text{ on } (0, 1]. \]

Solution:
\[
\int_0^1 x^{-p} dx = \begin{cases} 
\frac{2}{p} x^{1-p} \bigg|_0^1 & \text{if } p \neq 2 \\
(\ln(x))' \bigg|_0^1 & \text{if } p = 2 
\end{cases}
\]

\[ \Rightarrow 1 \leq p < 2 \]

\[ \text{iv). } f(x) = x \log(x), \text{ on } (0, 3]. \]

Solution:
\[
\int_0^1 x \log(x) dx = \left. x \log(x) - x \right|_0^1 = -1.
\]

Assume \( p \) is an integer. Therefore, integrating by parts:
\[
\int_0^1 x \log(x)^p dx = \left. x \log(x)^p \right|_0^1 - \int_0^1 \log(x)^p dx
\]
\[
= -\int_0^1 \log(x)^{p-1} dx.
\]

It follows by induction that \( \int_0^1 x \log(x)^p dx \) is finite. Since, \( \int_0^1 x \log(x)^p dx \) is monotone in \( p \) it follows that \( \int_0^1 x \log(x)^p dx \) is finite for all \( 1 \leq p < \infty \).

\[ \text{v). } f(x) = x^{1/2} \text{ on } (0, \infty). \]

Solution:
\[
\int_0^\infty x^{-p} \frac{1}{2} dx = \begin{cases} 
\frac{2}{p} x^{1-p} \bigg|_0^\infty & \text{if } p \neq 2 \\
(\ln(x))' \bigg|_0^\infty & \text{if } p = 2 
\end{cases}
\]

Consequently, for all \( p \in \mathbb{C}, \mathbb{C} \) \( \int_0^\infty x^{-p} dx \) is divergent.

\[ \text{vi). } f(x) = \begin{cases} 
x^{1/2} & 0 < x \leq 1 \\
2x^2 & 1 < x < \infty
\end{cases} \]

Solution:
\[
\int_0^\infty |f(x)|^p dx = \int_0^1 x^{-p/2} dx + 2^p \int_1^\infty x^{-2p} dx
\]

Now,
\[
\int_1^\infty x^{-2p} dx = \frac{x^{-2p+1}}{-2p+1} \bigg|_1^\infty < \infty.
\]

Therefore, by item \( \text{ii} \) it follows that \( 1 \leq p < 2 \).
C.) Is there a function $f$ on $(0,\infty)$ which belongs to $L^1_0$ but not $L^2_0$? Is there one that belongs to $L^2_0$ but not $L^1_0$?

Solution:
Let $f(t) = \begin{cases} \frac{1}{t} & 0 < t \leq 1, \\
0 & 1 < t < \infty \end{cases}$ Then $f \in L^1_0$ but not $L^2_0$.

Let $g(t) = \begin{cases} 0 & 0 < t \leq 1, \\
\frac{1}{t} & 1 < t < \infty \end{cases}$ Then $g \in L^2_0$ but not $L^1_0$.

d.) Is there a function $f$ on $(0,1)$ which belongs to $L^1_0$ but not to $L^2_0$? Is there one that belongs to $L^2_0$ but not $L^1_0$?

Solution:
Let $f(x) = \frac{1}{\sqrt{x}}$. Then $f \in L^1_0$.

Suppose $g \in L^2_0$. Then, by Hölder's inequality:

$$\int_0^1 |g(x)| dx \leq \left( \int_0^1 g(x)^2 dx \right)^{\frac{1}{2}} \left( \int_0^1 dx \right)^{\frac{1}{2}} = \|g\|_{L^2_0} < \infty.$$

Consequently, $g \in L^1_0$. 