Homework #6

#1.

a) Prove that:

$$\int_1^\infty \frac{e^{-t}}{t}\,dt \le e^{-1}$$

<u>Solution</u>:

On $[1,\infty)$ $t^{-1} \le 1$ and therefore,

$$\int_1^\infty \frac{e^{-t}}{t}\,dt \le \int_1^\infty e^{-t}\,dt = e^{-1}$$

b) Prove the better bound:

$$\int_1^\infty \frac{e^{-t}}{t}\,dt \le \frac{e^{-1}}{\sqrt{2}}$$

<u>Solution</u>:

By Hölder's inequality

$$\int_1^\infty \frac{e^{-t}}{t}\,dt \le \left(\int_1^\infty e^{-2t}\,dt\right)^{1/2}\left(\int_1^\infty \frac{1}{t^2}\,dt\right)^{1/2}$$

$$= \left(-\frac{1}{2}e^{-2t}\Big|_1^\infty\right)^{1/2}\left(-\frac{1}{t}\Big|_1^\infty\right)^{1/2}$$

$$= \frac{e^{-1}}{\sqrt{2}}.$$

c) Let $f \in L_0^p([a,b])$, $g \in L_0^q([a,b])$, $h \in L_0^r([a,b])$, where $1/p + 1/q + 1/r = 1$.

Prove that

$$\int_a^b |f(t)g(t)h(t)|\,dt \le \|f\|_{L^p}\cdot\|g\|_{L^q}\cdot\|h\|_{L^r}.$$

proof:

$$\int_a^b |f(t)g(t)h(t)|\,dt \le \left(\int_a^b |f(t)|^p dt\right)^{1/p}\left(\int_a^b |g(t)h(t)|^{\frac{rq}{r+q}} dt\right)^{\frac{r+q}{rq}}$$

$$\le \left(\int_a^b |f(t)|^p dt\right)^{1/p}\left(\int_a^b |g(t)|^{\frac{rq}{r+q}}\cdot|h(t)|^{\frac{rq}{r+q}}\right)^{\frac{r+q}{rq}}$$

Since, $\frac{q}{r+q} + \frac{r}{r+q} = 1$ it follows that:

$$\int_a^b |f(t)g(t)h(t)|\,dt \le \|f\|_{L^p}\left(\int_a^b |g(t)|^r\right)^{\frac{r+q}{rq}\cdot\frac{q}{r+q}}\left(\int_a^b |h(t)|^q dt\right)^{\frac{r+q}{rq}\cdot\frac{r}{r+q}}$$

$$= \|f\|_{L^p}\cdot\|g\|_{L^r}\cdot\|h\|_{L^q}.$$

d.) Let $f \in L^2_0([0,\pi])$. Is it possible to have simultaneously:

$$\int_0^\pi (f(t)-\sin(t))^2\,dt \le 1/4 \quad \text{and} \quad \int_0^\pi (f(t)-\cos(t))^2\,dt \le 1/4.$$

Solution:

No. If this was true then

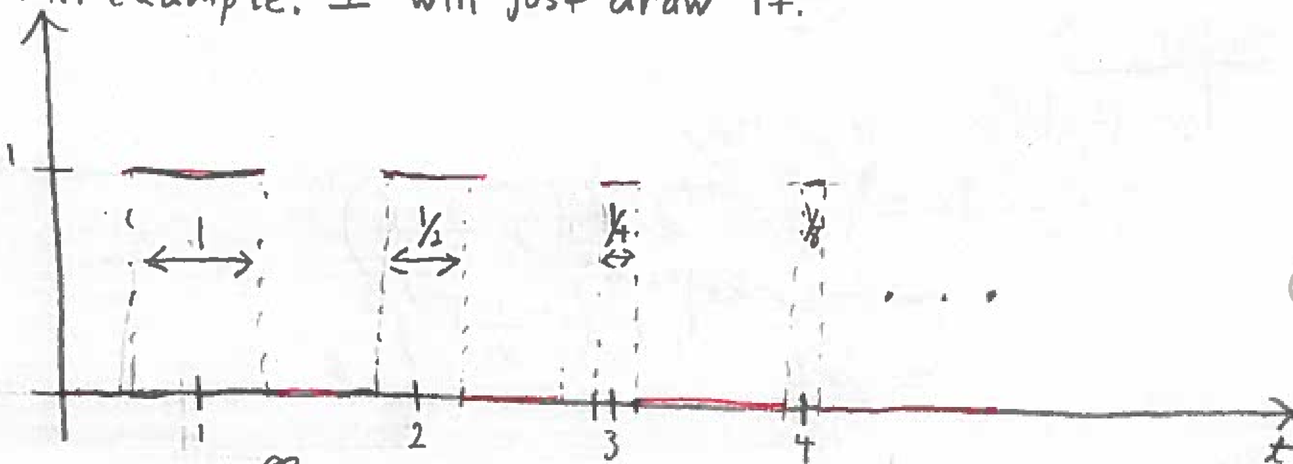
$$\sqrt{\pi} = \|\sin(t)-\cos(t)\|_{L^2_0(0,\pi)} \le \|f-\sin(t)\|_{L^2(0,\pi)} + \|f-\cos(t)\|_{L^2(0,\pi)} \le 1.$$

e.) Suppose that $\int_0^\infty |f(t)|\,dt < \infty$. Prove or give a counterexample:

$$\lim_{t\to\infty} |f(t)| = 0.$$

Solution:

Counterexample. I will just draw it.



$$\int_0^\infty f(x)\,dx = \sum_{n=0}^\infty \frac{1}{2^n} < \infty.$$

However, $\lim_{x\to\infty} f(x) \ne 0$.

#2.

f is continuously differentiable function such that $f(0)=f(L)=0$ and its energy norm is defined by

$$\|f\|_E = \left(\int_0^L (f'(x))^2 dx \right)^{1/2}.$$

Prove that

$$\|f\|_p \leq \left(\frac{2}{p+2}\right)^{1/p} \left(L^{1+p/2}\right)^{1/p} \|f\|_E.$$

proof:

$$f(x) = \int_0^x f'(t)\,dt$$

$$\Rightarrow |f(x)| = \left| \int_0^x f'(t)\,dt \right| \leq \int_0^x |f'(t)|\,dt$$

Applying Hölder's inequality

$$\int_0^x |f'(t)|\,dt \leq \left(\int_0^x 1^2\,dt \right)^{1/2} \left(\int_0^x |f'(t)|^2\,dt \right)^{1/2}$$
$$= x^{1/2} \|f\|_E.$$

Therefore,

$$\|f\|_p \leq \|f\|_E \left(\int_0^L x^{p/2}\,dx \right)^{1/p}$$
$$= \|f\|_E \cdot \left(\frac{L^{p/2+1}}{p/2+1} \right)^{1/p}$$
$$= \|f\|_E \cdot \left(L^{\frac{p+2}{2p}} \right) \cdot \left(\frac{2}{p+2} \right)^{1/p}$$

#3.

a.) Let $f_n \in C([0,1])$ be an equicontinuous sequence of functions. If $f_n \to f$ pointwise, prove that f is continuous.

proof:

By triangle inequality it follows that

$$|f(x)-f(y)| \leq |f_n(x)-f(x)| + |f_n(x)-f_n(y)| + |f_n(y)-f(y)|$$

Since $f_n \to f$ pointwise there exists $N \in \mathbb{N}$ such that $n \geq N(x,y)$ $|f_n(x)-f(x)| < \varepsilon$ and $|f_n(y)-f(y)| < \varepsilon$. Since f_n is equicontinuous there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f_n(x)-f_n(y)| < \varepsilon$. Therefore, for $|x-y| < \delta$ if $n \geq N(x,y)$ it follows that

$$|f(x)-f(y)| < 3\varepsilon.$$

b.) Let $K \subset \mathcal{C}([0,1])$ be defined by

$$K = \{ f \in \mathcal{C}([0,1]) : Lip(f) \leq 1 \text{ and } \int_0^1 f(x)dx = 0 \}$$

Prove that K is compact in $\mathcal{C}([0,1])$ with respect to $\|\cdot\|_\infty$.

~~proof~~:

Let f_n be a sequence in K.

1. Since $Lip(f_n) \leq 1$ it follows that for all $x, y \in [0,1]$:

$$|f_n(x) - f_n(y)| \leq |x-y|.$$

Consequently, f_n is equicontinuous.

2. For contradiction suppose f_n is unbounded. Therefore, there exists $a_n \in [0,1]$ such that $\max|f_n| = |f(a_n)|$. Since $\int_0^1 f(x)dx = 0$ it follows that there exists $b_n \in [0,1]$ such that $f(b_n) = 0$ and f does not change sign on $[a_n, b_n]$ if $a_n < b_n$ or $[b_n, a_n]$ if $b_n < a_n$. Therefore,

$$1 \geq Lip(f) = \sup_{x,y} \frac{|f(x)-f(y)|}{|x-y|} \geq \frac{|f(a_n)|}{|b_n - a_n|} \geq \frac{|f(a_n)|}{1}$$

Which is a contradiction.

By items 1 and 2 K is compact by Arzela–Ascoli.

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#4.

a.) Consider the following scalar differential equation:

$$\frac{du}{dt} = |u(t)|^{\alpha},$$
$$u(0) = 0.$$

Show that the solution is unique if $\alpha \geq 1$, but not if $0 \leq \alpha < 1$.

Solution:

1. If $0 < \alpha < 1$, we can formally solve this equation:

$$\int_0^{u(t)} \frac{1}{u^{\alpha}} \, du = t$$

$$\Rightarrow \frac{u^{1-\alpha}}{1-\alpha} = t$$

$$\Rightarrow u(t) = [(1-\alpha)t]^{1/1-\alpha}.$$

However, we can construct a pieceswise solution:

$$u(t) = \begin{cases} 0 & t \leq a \\ [(1-\alpha)(t-a)]^{1/1-\alpha} & t > a \end{cases}.$$

2. If $\alpha \geq 1$, then $|u(t)|^{\alpha}$ is Lipshitz and hence the solution is unique.

b.) Suppose that $f(t,u)$ is a continuous function $f: \mathbb{R}^2 \to \mathbb{R}$ such that for all $t, u, v \in \mathbb{R}$:

$$|f(t,u) - f(t,v)| \leq K|u - v|.$$

Also suppose that

$$M = \sup\{|f(t, u_0)|; |t - t_0| \leq T\}.$$

Prove that the solution of the inital value problem

$$\frac{du}{dt} = f(t,u)$$
$$u(t_0) = u_0$$

satisfies the estimate

$$|u(t) - u_0| \leq MTe^{KT}$$

for $|t - t_0| \leq T$.

Solution:

Integrating it follows that

$$u(t)-u_0=\int_{t_0}^{t} f(s,u(s))ds$$

$$=\int_{t_0}^{t}[f(s,u(s))-f(s,u_0)]ds+\int_{t_0}^{t} f(s,u_0)ds$$

$$\Rightarrow |u(t)-u_0|\le\int_{t_0}^{t}|f(s,u(s))-f(s,u_0)|ds+\int_{t_0}^{t}|f(s,u_0)|ds$$

$$\le\int_{t_0}^{t}K|u(s)-u_0|ds+\int_{t_0}^{t}M$$

$$\le\int_{t_0}^{t}K|u(s)-u_0|ds+MT.$$

Applying Gronwall's inequality it follows that:

$$|u(t)-u_0|\le MT\exp\left(\int_{t_0}^{t}K\,ds\right)\le MT\exp(KT).$$