Homework 8

Analysis

Due: March 26, 2018

1. **Convolutions:**

   (a) Let f, g be smooth functions with compact support. Let A be the closure of the set

   $$\{x + y : x \in \operatorname{supp}(f) \text{ and } y \in \operatorname{supp}(g)\}$$

   Prove that $\operatorname{supp}(f * g) \subset A$.

   (b) Draw a picture of a smooth function f on $\mathbb{R}$ satisfying

   - f has compact support.
   - For all $x \in \mathbb{R}$, $0 \leq f(x) \leq 1$.

   Draw a picture of $f * f$.

   (c) Let $f = \chi_{[-1,1]}$. Find $f * f$ without calculating anything. I.e, try to just draw $f * f$ to obtain the formula for $f * f$.

   (d) Let $f, g \in C(\mathbb{R})$ be smooth functions with compact support. Prove that

   $$\|f * g\|_\infty \leq \|f\|_{L^p} \|g\|_{L^q},$$

   where $\frac{1}{p} + \frac{1}{q} = 1$.

2. **Equivalence Classes:**

   (a) In a metric space (M, d), say that $x \sim y$ if $d(x, y) < 1$. Is this an equivalence relation?

   (b) Let X be the set of 2×2 complex valued matrices. Say that $A \sim B$ if $B = CAC^{-1}$ for some invertible matrix C. Prove that $\sim$ is an equivalence relation. Prove that the function $f(A) = \operatorname{trace}(A)$ is defined unambiguously on the set of equivalence classes as well.

3. **Abstract Completions:**

   (a) Recall, a metric space (X, d) is called bounded if there is a $K > 0$ such that $d(x, y) \leq K$ for all $x, y \in X$. Let (X, d) be bounded and suppose that (X, d) and (X', d') are isometric. Show that (X', d') is bounded.

   (b) Let (X, d) be metric space with completion $(\tilde{X}, \tilde{d})$. Suppose that (X', d') is a complete metric space, and suppose that there is an isometry $F : X \mapsto X'$ whose range $F(X)$ is dense in X'. Prove that (X', d') and $(\tilde{X}, \tilde{d})$ are isometric.

   (c) Prove that

   $$d(x, y) = \frac{|x - y|}{\sqrt{(1 + x^2)(1 + y^2)}}$$

   defines a metric on $\mathbb{R}$. Show that $\mathbb{R}$ is not complete in this metric. Find the completion.

Homework #8

#1.

a.) Let f, g be smooth functions with compact support. Let A be the closure of the set

$$\{x+y : x \in \text{supp}(f) \text{ and } y \in \text{supp}(g)\}.$$

Prove that $\text{supp}(f*g) \subset A$.

~~proof~~:

First,

$$\{x+y : x \in \text{supp}(f) \text{ and } y \in \text{supp}(g)\} = \{z ; z-y \in \text{supp}(f) \text{ and } y \in \text{supp}(g)\}$$

$$\Rightarrow \{x+y : x \in \text{supp}(f) \text{ and } y \in \text{supp}(g)\} = \{x : x-y \in \text{supp}(f) \text{ and } y \in \text{supp}(g)\}.$$

Now, if

$(f*g)(x) \neq 0$ then

$$\int_{-\infty}^{\infty} f(x-y)g(y)\,dy \neq 0$$

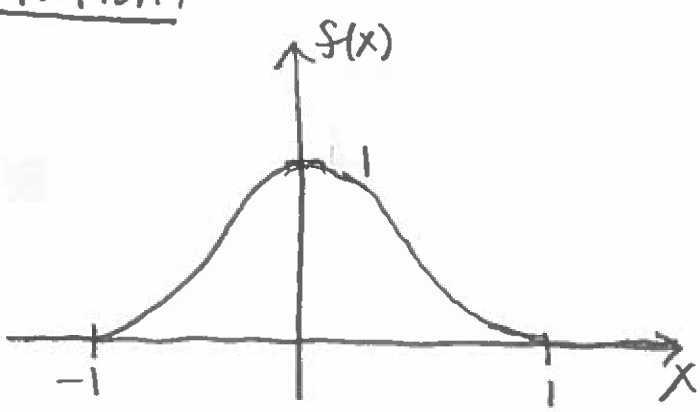
Therefore, $x-y \in \text{supp}(f)$ and $y \in \text{supp}(g)$.

b.) Draw a picture of a function f on $\mathbb{R}$ satisfying

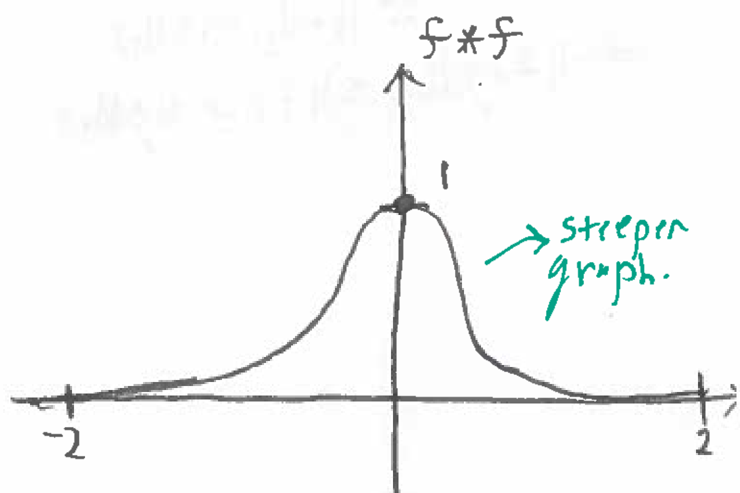
- f has compact support
- For all $x \in \mathbb{R}$, $0 \leq f(x) \leq 1$.

Draw a picture of $f * f$.

Solution:



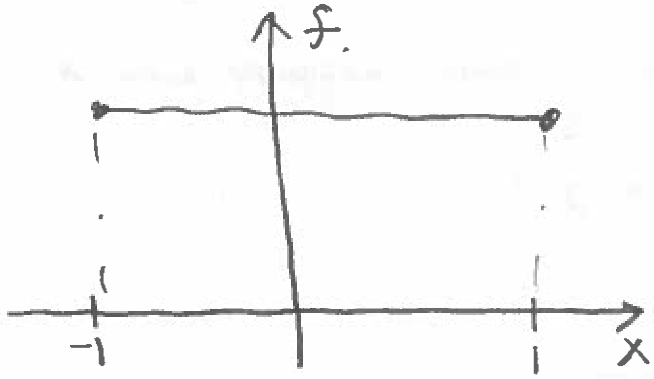
$\text{supp}(f) = (-1, 1)$

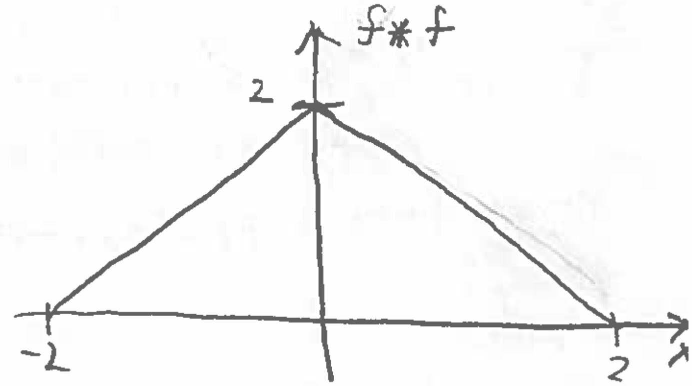


$\text{supp}(f*f) = (-2, 2)$

c.) Let $f = \chi_{[-1,1]}$. Find $f * f$ without calculating anything.

Solution:





d.) Let $f, g \in C^{\infty}(\mathbb{R})$ be smooth functions with compact support. Prove that

$$\|f * g\|_{\infty} \leq \|f\|_{L^p} \cdot \|g\|_{L^q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Solution:

$$\begin{aligned}
|f*g|(x) &= \left|\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\right| \\
&\leq \int_{-\infty}^{\infty} |f(x-y)|\cdot|g(y)|\,dy \\
&\leq \left(\int_{-\infty}^{\infty} |f(x-y)|^p\,dy\right)^{1/p} \left(\int_{-\infty}^{\infty} |g(y)|^q\,dy\right)^{1/q} \\
&= \left(\int_{-\infty}^{\infty} |f(y)|^p\,dy\right)^{1/p} \left(\int_{-\infty}^{\infty} |g(y)|^q\,dy\right)^{1/q} \\
&= \|f\|_{L^p} \cdot \|g\|_{L^q}
\end{aligned}$$

$$\Rightarrow \|f*g\|_{L^{\infty}} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}.$$

#2

a.) In a metric space (X, d), say that $x \sim y$ if $d(x,y) < 1$. Is this an equivalence relation?

Solution:

No, let $X = \mathbb{R}$ and $d(x,y) = |x-y|$. Then, $\frac{1}{4} \sim \frac{3}{4}$, $\frac{3}{4} \sim \frac{5}{4}$ but $\frac{1}{4}$ is not equivalent to $\frac{5}{4}$.

b.) Let X be the set of 2×2 complex valued matrices. Say that $A \sim B$ if $B = CAC^{-1}$ for some invertible matrix C. Prove that $\sim$ is an equivalence relation. Prove that the function $f(A) = \text{trace}(A)$ is defined unambiguously on the set of equivalence classes as well.

Solution:

1. Let $A \in \mathbb{C}^{2\times 2}$. Then $A = I A I^{-1} \Rightarrow A \sim A$.
2. Let $A, B \in \mathbb{C}^{2\times 2}$ and suppose $A \sim B$. Then, there exists $C \in \mathbb{C}^{2\times 2}$ such that

$$A = CBC^{-1}$$
$$\Rightarrow C^{-1}AC = B$$
$$\Rightarrow A \sim B.$$

3. Let $A, B, C \in \mathbb{C}^{2\times 2}$ and suppose $A \sim B$ and $B \sim C$. Then there exists $D, E \in \mathbb{C}^{2\times 2}$ such that

$$A = DBD^{-1} \text{ and } B = E \cdot CE^{-1}$$
$$\Rightarrow A = DECE^{-1}D^{-1} = (DE)C(DE)^{-1}.$$

Therefore, $A \sim C$.

Finally, $\text{trace}(A) = \lambda_1 + \lambda_2$, where λ_1, λ_2 are the eigenvalues of A. Since eigenvalues are preserved by similarity transformations the result follows.

#3.

a.) Let (X,d) be bounded and suppose (X,d) and (X',d') are isometric. Show that (X',d') is bounded.

Solution:

Since, (X',d') and (X,d) are isometric it follows that there exists $f: X' \to X$ such that for all $x,y \in X'$:

$$d'(x,y) = d(f(x), f(y)) \leq K.$$

b.) Let (X,d) be a metric space with completion $(\tilde{X},\tilde{d})$. Suppose that (X',d') is a complete metric space, and suppose there is an isometry $F: X \to X'$ whose range $F(X)$ is dense in X'. Prove that (X',d') and $(\tilde{X},\tilde{d})$ are isometric.

proof:

Define $\bar{F}: \tilde{X} \to X'$ by

$$\bar{F}([x_n]) = \lim_{n\to\infty} F(x_n).$$

Then,

$$d'(\bar{F}([x_n]), \bar{F}([y_n])) = d'(\lim_{n\to\infty} F(x_n), \lim_{n\to\infty} F(y_n))$$
$$= d(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n)$$
$$= \tilde{d}([x_n],[y_n]).$$

c.) Prove that

$$d(x,y) = \frac{|x-y|}{\sqrt{(1+x^2)(1+y^2)}}$$

defines a metric on $\mathbb{R}$. Show that $\mathbb{R}$ is not complete in this metric. Find the completion.

proof:

~~If~~ $x,y \in \mathbb{R}$ then there exists $\alpha, \beta \in (-\pi/2, \pi/2)$ such that

$$\tan(\alpha) = x$$
$$\tan(\beta) = y$$

Therefore,

$$d(x,y) = \frac{|\tan(\alpha) - \tan(\beta)|}{\sqrt{1+\tan^2(\alpha)}\sqrt{1+\tan^2(\beta)}} = \frac{|\tan(\alpha) - \tan(\beta)|}{|\sec(\alpha)\sec(\beta)|} = |\sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha)|$$

$$\Rightarrow d(x,y) = |\sin(\alpha - \beta)|$$

Triangle inequality follows from concavity of $|\sin(x)|$.

Now, if x_n is Cauchy with $\tan(\alpha_n) = x_n$ it follows that there exists $N \in \mathbb{N}$ such that $m,n > N$

$$\Rightarrow d(x_n, x_m) = |\sin(\alpha_m - \alpha_n)| < \varepsilon.$$

For this to be true it follows from continuity of sin that

$$\alpha_m - \alpha_n \to 0 \quad \text{or} \quad \alpha_m - \alpha_n \to \pi.$$

Therefore, since $x_n = \tan^{-1}(\alpha_n)$ it follows ~~from~~ continuity that

1. $x_n \to x \in \mathbb{R}$
2. $x_n \to \pm\infty$.

However, with this metric all unbounded sequences are equivalent. Therefore, the completion is the standard completion with equivalence class of unbounded sequences included

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