Lecture #1: Review of 611 via examples

What is analysis?
- Rigorous foundation of calculus (narrow view)
- Rigorous approximations (numerical view)
- Mathematical study of limiting processes (mature view)
- Applications - probability, differential equations, physics, dynamical systems, applied math.

My view:
- Start with hard problem.
- Approximate hard problem with simple problem that can be solved exactly.
- Take limit.
- Analysis is the necessary glue needed to ensure this process works,

**Example 1:**
How do you find slope of tangent line

\[ f'(x) = \text{limit of secant lines.} \]
Example 2:
Area under curve

\[ f_n \rightarrow \text{Function with } n \text{ rectangles.} \]

\[ f = \cdots \]

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\[ \lim_{n \to \infty} A(f_n) = \int_a^b f(x) \, dx, \quad A \text{ is the area function.} \]

Example 3:
\[ \sqrt{2} = 2. \]

Proof:
Let L denote the length of a curve

\[ = x_1, \quad L(x_1) = 2 = a_1 \]

\[ \frac{1}{2} = x_2, \quad L(x_2) = 2 = a_2 \]

\[ \frac{1}{2} \]

\[ = x_{\infty}, \quad L(x_{\infty}) = \sqrt{2} \]
\[
\sqrt{2} = L(\alpha_{\infty}) = L(\lim_{n \to \infty} \alpha_n) = \lim_{n \to \infty} L(\alpha_n) = \lim_{n \to \infty} 2 = 2.
\]

The proof is obviously wrong:
\[
L(\lim_{n \to \infty} \alpha_n) \neq \lim_{n \to \infty} L(\alpha_n).
\]

We only knew it was wrong because we knew the answer already.

Let's take a deeper look at this example.
\[
\alpha_n(t) = (x_n(t), y_n(t)), \quad \text{(parametric form of curve)}
\]

\[
\Rightarrow L(\alpha_n(t)) = \int_0^2 \sqrt{x'(t)^2 + y'(t)^2} \, dt
\]

The functions \( x_n', y_n' \) converge to infinite wiggly function.

The strange fact is
\[
\lim_{t \to \infty} x_n(t) = \frac{1}{2}t, \quad \lim_{t \to \infty} y_n(t) = \frac{1}{2}t
\]

But:
\[
\lim_{t \to \infty} x_n'(t) \neq \frac{1}{2}, \quad \lim_{t \to \infty} y_n'(t) \neq \frac{1}{2} \quad \text{(limit does not exist)}
\]
Example 4:
Consider a function $f: \mathbb{R} \to \mathbb{R}$. Under what conditions does $f$ have a minimum?

Pictures:

Is continuity necessary? No.

A function $f: \mathbb{R} \to \mathbb{R}$ is lower semicontinuous if for all $x \in \mathbb{R}$ and every sequence $x_n \to x$, we have

$$f(x) \leq \liminf_{n \to \infty} f(x_n)$$

Recall:
1. Let $b_n = \sup \{x_k : k \geq n \}$. Then
   $$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} b_n = \lim_{n \to \infty} \sup \{x_k : k \geq n \}$$

2. Let $a_n = \inf \{x_k : k \geq n \}$
   $$\liminf_{n \to \infty} x_n = \lim_{n \to \infty} a_n = \lim_{n \to \infty} \inf \{x_k : k \geq n \}.$$
Definition - A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is coercive if
\[ \lim_{|x| \to \infty} f(x) = \infty \]

*Explicitly*: For any $M > 0$, there exists $R > 0$ such that $|x| > R$ implies $f(x) = M$.

![Diagram](image)

Theorem - If $f : \mathbb{R} \rightarrow \mathbb{R}$ is lower semicontinuous and coercive, then there exists $x^*$ such that for all $x \in \mathbb{R}$:
\[ f(x^*) \leq f(x). \]

Proof:
Let $x_n$ be a minimizing sequence of $f$. I.e.
\[ \lim_{n \to \infty} f(x_n) = \inf_{x \in \mathbb{R}} f(x). \]

Since $f$ is coercive there exists $R > 0$ such that for all $n \in \mathbb{N}$, $|x_n| < R$. Therefore, by Bolzano-Weierstrass there exists $x^* \in [-R, R]$ and a subsequence $x_{n_k}$ such that
\[ \lim_{k \to \infty} x_{n_k} = x^*. \]

Therefore,
\[ f(x^*) \leq \liminf_{k \to \infty} f(x_{n_k}) = \lim_{n \to \infty} f(x_n) = \inf_{x \in \mathbb{R}} f(x). \]