Lecture 12: Contraction Mapping Theorem

Definition - Let \((X, d)\) be a metric space. A mapping \(T: X \to X\) is a contraction mapping if there exists a constant \(c\), with \(0 \leq c < 1\) such that

\[ d(T(x), T(y)) \leq c d(x, y) \]

for all \(x, y \in X\).

\[ \text{Definition - If } T: X \to X, \text{ then a point } x \in X \text{ such that } T(x) = x \]

is called a fixed point of \(T\).

Theorem (Contraction Mapping) - If \(T: X \to X\) is a contraction mapping on a complete metric space \((X, d)\), then there is a unique fixed point of \(T\).

\[ \text{Proof:} \]

Let \(x_0 \in X\). Define a sequence by

\[ x_{n+1} = T x_n. \]

Denote the \(n\)-th composition of \(T\) by \(T^n\). Therefore,

\[ x_n = T^n x_0. \]
Now,
\[
d(x_n, x_0) = d(T^n x_0, T^m x_0)
\]
\[
= d(T^n x_0, T^n (T^{m-n} x_0))
\]
\[
\leq c^n d(x_0, T^{m-n} x_0)
\]
\[
\leq c^n [d(x_0, Tx_0) + d(Tx_0, T^{m-n} x_0)]
\]
\[
= c^n \left[ d(x_0, x_1) + d(Tx_0, T^{m-n} x_0) \right]
\]
\[
\leq c^n \left[ d(x_0, x_1) + d(Tx_0, T^2 x_0) + d(T^2 x_0, T^{m-n} x_0) \right]
\]
\[
\leq c^n \left[ d(x_0, x_1) + c d(x_0, x_1) + d(Tx_0, T^{m-n} x_0) \right]
\]
\[
\leq c^n \left[ \sum_{k=0}^{m-n-1} c^k \right] d(x_0, x_1)
\]
\[
\leq c^n \left[ \sum_{k=0}^{\infty} c^k \right] d(x_0, x_1)
\]
\[
\leq \frac{c^n d(x_1, x_0)}{1-c}
\]

Therefore, \( x_n \) is Cauchy. By completeness there exists \( x^* \) such that \( x_n \xrightarrow{\text{in X}} x^* \in X \). Let \( T \) be continuous and thus
\[
Tx^* = T \lim_{n \to \infty} x_n = \lim_{n \to \infty} Tx_0 = \lim_{n \to \infty} x_{n+1} = x^*.
\]

Finally, if \( x^*, y^* \) are two fixed points, then
\[
0 \leq d(x^*, y^*) = d(Tx^*, Ty^*) \leq c d(x^*, y^*).
\]
\[
\Rightarrow d(x^*, y^*) = 0.
\]
Example: (Integral Equations)

Let \( K(x,y) \) be a smooth function on \( \mathbb{R}^2 \times \mathbb{R}^2 \)\\
Under what conditions on \( \lambda \) does\\
\[ u(x) = f(x) + \lambda \int_a^b K(x,y) u(y) \, dy \]

have a continuous solution \( u(x) \)?

Define \( T: C([a,b]) \to C([a,b]) \) by:
\[ T(u(x)) = f(x) + \lambda \int_a^b K(x,y) u(y) \, dy \]

Now,
\[ |T(u(x)) - T(v(x))| = | \lambda \int_a^b K(x,y) (u(y) - v(y)) \, dy | \]
\[ \leq |\lambda| \int_a^b |K(x,y)| |u(y) - v(y)| \, dy \]
\[ \leq |\lambda| \|u - v\|_{\infty} \int_a^b |K(x,y)| \, dy \]

\[ \Rightarrow \|Tu - Tv\| \leq |\lambda| \max_x \int_a^b |K(x,y)| \, dy \cdot \|u - v\|_{\infty} \]

Therefore, if
\[ |\lambda| < \frac{1}{\max_x \int_a^b |K(x,y)| \, dy} \]

it follows from the contraction mapping theorem that \( T \) has a fixed point \( u^* \). Consequently,
\[ u^*(x) = f(x) + \lambda \int_a^b K(x,y) u^*(y) \, dy \]
Example (Linear ODEs):

\[
\begin{align*}
\frac{d}{dt} \vec{u}(t) &= \vec{A}(t) \vec{u}(t) + \vec{b}(t) \\
\vec{u}(0) &= \vec{U}_0
\end{align*}
\]

where \( \vec{A}: \mathbb{R} \rightarrow \mathbb{R}^{n \times n} \) is a continuous matrix-valued function, \( \vec{b}: \mathbb{R} \rightarrow \mathbb{R}^n \) is a continuous vector-valued function.

Is there a solution to this equation?

\[ \Rightarrow \vec{u}(t) = \vec{U}_0 + \int_0^t [\vec{A}(s) \vec{u}(s) + \vec{b}(s)] ds \]

Define \( T \) by 

\[ T \vec{v} = \int_0^t [\vec{A}(s) \vec{v}(s) + \vec{b}(s)] ds \]

\[ \Rightarrow \| T \vec{v} - T \vec{w} \|_\infty \leq \int_0^t \| [\vec{A}(s) \vec{v}(s) - \vec{A}(s) \vec{w}(s)] \|_\infty ds \]

\[ \leq \int_0^t \| \vec{A}(s) \|_\infty \| \vec{v}(s) - \vec{w}(s) \|_\infty ds \]

\[ = \| \vec{A} \|_\infty \| \vec{v}(s) - \vec{w}(s) \|_\infty \cdot t \]

Therefore, for \( t < \frac{1}{\| \vec{A} \|_\infty} \), the contraction mapping theorem guarantees a unique solution.

A global solution can be constructed using overlapping intervals.
Example (Discrete Dynamical Systems):

\[ X_{n+1} = \sqrt{X_n} = f(X_n) \]
\[ X_0 = a \geq 0 \]

Fixed points are given by \( x^* = 0 \) and \( x^* = 1 \).

\( x^* = 1 \) is stable, meaning nearby starting initial conditions satisfy \( \lim_{n \to \infty} X_n = 1 \). How do we prove this?

\[ |x_{n+1} - x^*| = |\sqrt{x_n} - \sqrt{x^*}| = |\sqrt{x_n} - \sqrt{x^*}| \cdot |x_0 - x^*| \]

where \( c \in B_{x^*}(1) \). Therefore, if \( |f'(1)| < 1 \) we can select \( x_0 \) sufficiently small so that:

\[ |x_t - 1| < M |x_0 - 1|, \]

with \( M < 1 \). We can iterate this process to obtain:

\[ |x_n - 1| < M^n |x_0 - 1| \]

\[ \Rightarrow \lim_{n \to \infty} x_n = 1 \]

The key condition to show stability is that \( |f'(x^*)| < 1 \).