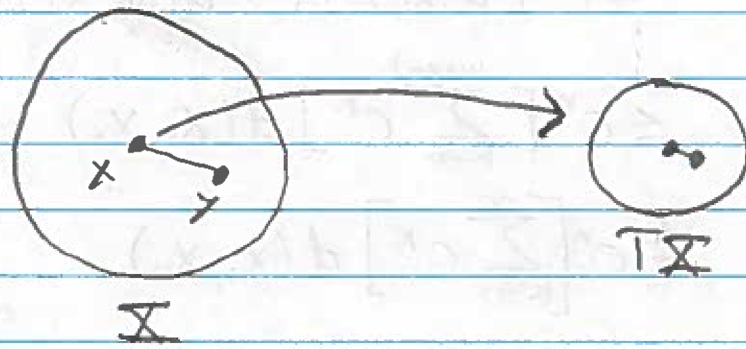
Lecture 12: Contraction Mapping Theorem

Definition - Let $(\mathfrak{X}, d)$ be a metric space. A mapping $T: \mathfrak{X} \to \mathfrak{X}$ is a contraction mapping if there exists a constant c, with $0 \leq c < 1$ such that

$$d(T(x), T(y)) \leq d(x,y)$$

for all $x, y \in \mathfrak{X}$.



Definition - If $T: \mathfrak{X} \to \mathfrak{X}$, then a point $x \in \mathfrak{X}$ such that

$$T(x) = x$$

is called a fixed point of T.

Theorem (Contraction Mapping) - If $T: \mathfrak{X} \to \mathfrak{X}$ is a contraction mapping on a complete metric space $(\mathfrak{X}, d)$, then there is a unique fixed point of T.

proof:

Let $x_0 \in \mathfrak{X}$. Define a sequence by

$$x_{n+1} = Tx_n.$$

Denote the n-th composition of T by T^n.

Therefore, $x_n = T^n x_0$.

Now,

$$
\begin{aligned}
d(x_n, x_m) &= d(T^n x_0, T^m x_0) \\
&= d(T^n x_0, T^n(T^{m-n} x_0)) \\
&\le c^n d(x_0, T^{m-n} x_0) \\
&\le c^n [d(x_0, Tx_0) + d(Tx_0, T^{m-n} x_0)] \\
&= c^n [d(x_0, x_1) + d(Tx_0, T^{m-n} x_0)] \\
&\le c^n [d(x_0, x_1) + d(Tx_0, T^2 x_0) + d(T^2 x_0, T^{m-n} x_0)] \\
&\le c^n [d(x_0, x_1) + c\, d(x_0, x_1) + d(T^2 x_0, T^{m-n} x_0)] \\
&\ \vdots \\
&\le c^n \left[\sum_{k=0}^{m-n-1} c^k\right] d(x_1, x_0) \\
&\le c^n \left[\sum_{k=0}^{\infty} c^k\right] d(x_1, x_0) \\
&\le \frac{c^n}{1-c} d(x_1, x_0)
\end{aligned}
$$

Therefore, x_n is Cauchy. By completeness there exists x^* such that $x_n \to x^* \in X$. Clearly T is continuous and thus

$$T x^* = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} T x_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

Finally, if x^*, y^* are two fixed points, then

$$0 \le d(x^*, y^*) = d(Tx^*, Ty^*) \le c\, d(x^*, y^*).$$

$$\Rightarrow d(x^*, y^*) = 0.$$

∎

Example (Integral Equations):

Let $K(x,y)$ be a smooth function on $[a,b]\times[a,b]$. Under what conditions on λ does

$$u(x) = f(x) + \lambda \int_a^b K(x,y)u(y)\,dy$$

have a continuous solution $u(x)$?

Define $T: C([a,b]) \to C([a,b])$ by:

$$T(u(x)) = f(x) + \lambda \int_a^b K(x,y)u(y)\,dy$$

Now,

$$|T(u(x)) - T(v(x))| = \left|\lambda \int_a^b K(x,y)(u(y)-v(y))\,dy\right|$$

$$\leq |\lambda| \int_a^b |K(x,y)|\,|u(y)-v(y)|\,dy$$

$$\leq |\lambda| \cdot \|u-v\|_\infty \int_a^b |K(x,y)|\,dy$$

$$\Rightarrow \|Tu - Tv\| \leq |\lambda| \max_x\left\{\int_a^b |K(x,y)|\,dy\right\} \cdot \|u-v\|_\infty$$

Therefore, if

$$|\lambda| < \frac{1}{\max_x\left\{\int_a^b |K(x,y)|\,dy\right\}}$$

it follows from the contraction mapping theorem that T has a fixed point u^*. Consequently,

$$u^*(x) = f(x) + \lambda \int_a^b K(x,y)u^*(y)\,dy.$$

Example (Linear ODEs):

$$\frac{d\vec{u}}{dt} = A(t)\vec{u}(t) + \vec{b}(t),$$
$$\vec{u}(0) = \vec{u}_0$$

where $A: \mathbb{R} \to \mathbb{R}^{n\times n}$ is a continuous matrix-valued function, $\vec{b}: \mathbb{R} \to \mathbb{R}^n$ is a continuous vector-valued function.

Is there a solution to this equation?

$$\Rightarrow \vec{u}(t) = \vec{u}_0 + \int_0^t [A(s)\vec{u}(s) + \vec{b}(s)]ds$$

Define T by $T\vec{u} = \int_0^t [A(s)\vec{u}(s) + \vec{b}(s)]ds$.

$$\Rightarrow \|T\vec{u} - T\vec{v}\|_\infty \leq \int_0^t \|A(s)(\vec{u}(s) - \vec{v}(s))\|_\infty ds$$
$$\leq \int_0^t \|A\|_\infty \|\vec{u}(s) - \vec{v}(s)\|_\infty ds$$
$$= \|A\|_\infty \cdot \|\vec{u}(s) - \vec{v}(s)\|_\infty \cdot t.$$

Therefore, for $t < 1/\|A\|_\infty$ the contraction mapping theorem guarantees a unique solution.

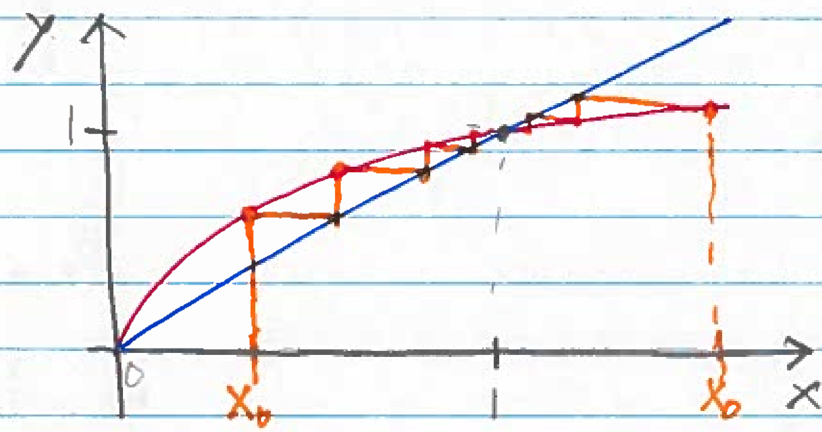
A global solution can be constructed using overlapping intervals.

Example (Discrete Dynamical Systems):

$$x_{n+1} = \sqrt{x_n} = f(x_n)$$

$$x_0 = a \geq 0$$

Fixed points are given by
$x^* = 0$ and $x^* = 1$.



$x^* = 1$ is stable, meaning nearby starting initial conditions satisfy $\lim_{n\to\infty} x_n = 1$. How do we prove this??

$$|x_1 - 1| = |f(x_0) - f(1)| = |f'(c)| \cdot |x_0 - 1|$$

Where $c \in B_{x_1}(1)$. Therefore, if $|f'(1)| < 1$ we can select x_1 sufficiently small so that:

$$|x_1 - 1| < M \cdot |x_0 - 1|,$$

with $M < 1$. We can iterate this process to obtain:

$$|x_n - 1| < M^n |x_0 - 1|$$

$$\Rightarrow \lim_{n\to\infty} x_n = 1.$$

The key condition to show stability is that
$|f'(x^*)| < 1$.