Lecture 15: Weak Convergence

**Definition:** Suppose that \(1 < p < \infty\). A sequence \(f_n\) converges weakly to \(f\) in \(L^p([a,b])\) written \(f_n \rightharpoonup f\), if

\[
\lim_{n \to \infty} \int_a^b f_n(x) g(x) \, dx = \int_a^b f(x) g(x) \, dx,
\]

for every \(g \in L^q\), \((\frac{1}{p} + \frac{1}{q} = 1)\).

**Example:**

a.) Let \(f_n(x) = \sin(nx)\). Then \(f_n \to 0\) on \([0,1]\) in \(L^2\).

b.) Let \(f \in L^2([0,1])\). Define \(f_n(x) = f(x-n)\). Then, \(f_n \to f\) in \(L^2\).

c.) Let \(f \in L^2\). Define \(f_n(x)\) by

\[
f_n(x) = \frac{1}{\sqrt{n}} f(nx).
\]

**Construction of weak topology**

A topology \(\tau\) on \(X\) is a collection of subsets satisfying

a.) \(\emptyset, X \in \tau\)

b.) If \(G_i \in \tau\) for \(i \in A\), the \(\bigcup_{i \in A} G_i \in \tau\)

c.) If \(G_i \in \tau\) for \(i = 1, \ldots, n\), then \(\bigcap_{i=1}^n G_i \in \tau\)

**Def.**

A function \(f : X \to Y\) between topological spaces \((X, \tau_1)\) and \((Y, \tau_2)\) is continuous if for all \(G \in \tau_2\), \(f^{-1}(G) \in \tau_1\).

**Goal:**

Let \(X\) be a linear space. Construct a topology in which all linear functionals are continuous.
Define \( G \in \mathcal{F} \) if \( G = f^{-1}(\text{open set in } \mathbb{R}) \) for some linear functional \( f \).

\[ L \hat{x} = a_1 x_1 + a_2 x_2 \quad \mathbb{R} \]

\[ \Rightarrow a < a_1 x_1 + a_2 x_2 < b \]

*In infinite dimensions this topology is weaker than the metric topology.*

**Definition:** Let \( X \) be a linear space and \( X^* \) its dual. A sequence \( x_n \in X \) converges weakly to \( x \) (written \( x_n \rightharpoonup x \)) if for all \( \varphi \in X^* \)

\[ \lim_{n \to \infty} \varphi(x_n) = \varphi(x). \]

**Examples:**
1. \( l^2^* = l^2 \). If \( x^{(m)} \in l^2 \) satisfies \( x^{(m)} \rightharpoonup x \) then

\[ \lim_{n \to \infty} \sum_{i=1}^{\infty} x^{(m)}_i y_i = \sum_{i=1}^{\infty} x_i y_i \quad \text{for all } y \in l^2. \]